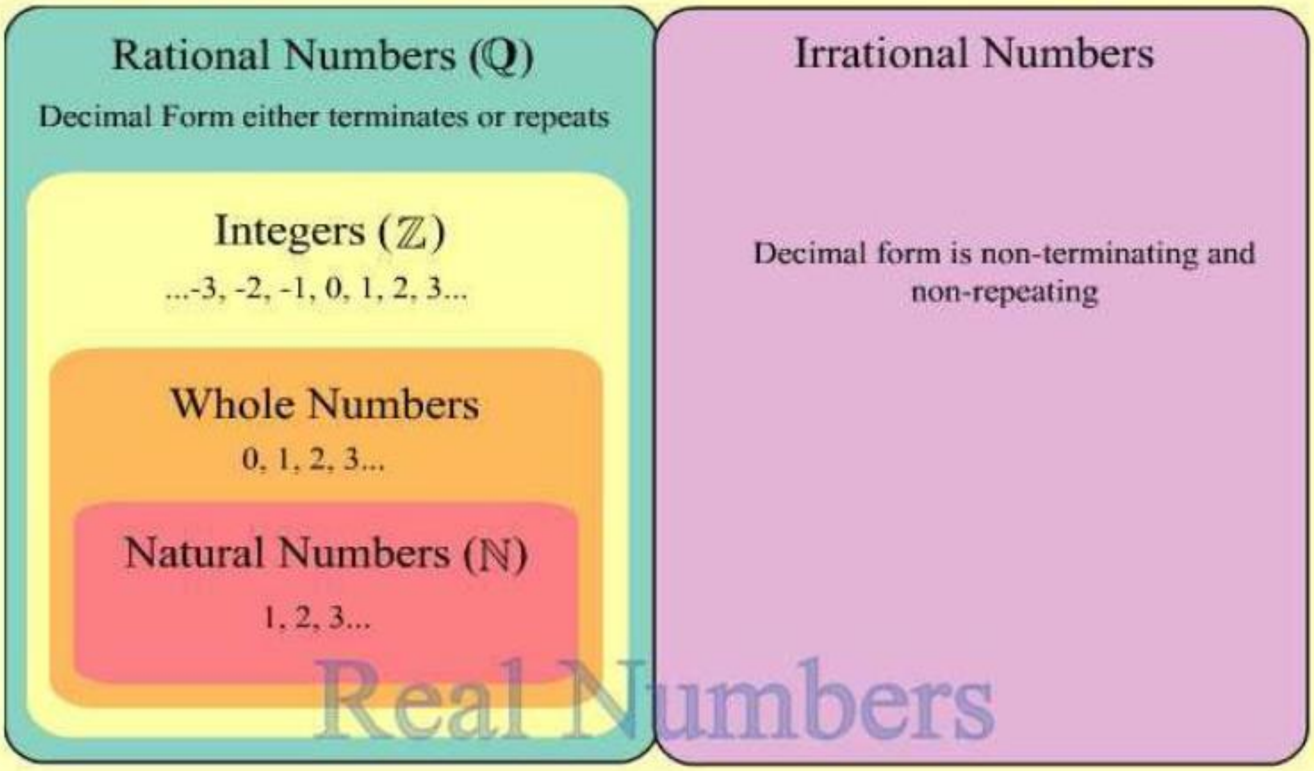
Week 01

Real Numbers and their properties

The following figure shows the relationships among the subsets of $\mathbb{R}$ (the real number system) defined earlier. This figure indicates, for example, that every natural number is automatically a whole number, and also an integer, and also a rational number.



Types of Real Numbers

The Natural (or Counting) Numbers: This is the set of numbers $\mathbb{N} = \{\ 1, 2, 3, 4, 5, \ldots\ \}$. The set is infinite, so in list form we can only write the first few such numbers.

The Whole Numbers: This is the set of natural numbers with 0 added: $\{\ 0, 1, 2, 3, 4, 5, \ldots\ \}$. Again, we can only list the first few members of this set. No special symbol will be assigned to this set in this course.

The Integers: This is the set of natural numbers, their negatives, and 0. As a list, this is the set $\mathbb{Z} = \{\ \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ \}$. Note that the list continues indefinitely in both directions.

The Rational Numbers: This is the set, with symbol $\mathbb{Q}$ for quotient, of *ratios* of integers (hence the name). That is, any rational number can be written in the form $\frac{p}{q}$, where p and q are both integers and $q \neq 0$. When written in decimal form, rational numbers either terminate or repeat a pattern of digits past some point.

The Irrational Numbers: Every real number that is not rational is, by definition, irrational. In decimal form, irrational numbers are non-terminating and non-repeating. No special symbol will be assigned to this set in this course.

The Real Numbers: Every set above is a subset of the set of real numbers, which is denoted $\mathbb{R}$. Every real number is either rational or irrational, and no real number is both.

Example 1 : Identifying Types of Real Numbers

In the following set identify the **a**. natural numbers, **b**. whole numbers, **c**. integers, **d**. rational numbers, and **e**. irrational numbers.

$$S = \{ -\sqrt{4}, \frac{5}{6}, 0, 2\sqrt{2}, 5.87, 2\sqrt{16}, 3\pi, 8^4 \}$$

Solution:

a. The natural numbers in S are $2\sqrt{16}$ and 8^4. $2\sqrt{16}$ is a natural number since $2\sqrt{16} = 8$.

b. The whole numbers in S are 0, $2\sqrt{16}$ and 8^4.

c. The integers in S are $-\sqrt{4}$, 0, $2\sqrt{16}$ and 8^4.

d. The rational numbers are $\frac{5}{6}$, 5.87, $-\sqrt{4}$, 0, $2\sqrt{16}$ and 8^4. Any integer p automatically qualifies as a rational number since it can be written as $\frac{p}{1}$.

e. The only irrational numbers in S are $2\sqrt{2}$ and 3π. Although well known now, the irrationality of $\sqrt{2}$ came as a bit of a surprise to the early Greek mathematicians who discovered this fact, and the irrationality of π was not proven until 1767.

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The Real number line

The Real Number Line

It is often convenient and instructive to depict the set of real numbers as a horizontal line, with each point on the line representing a unique real number and each real number associated with a unique point on the line. The real number corresponding to a given point is called the **coordinate** of that point. Thus, one (and only one) point on the line represents the number 0, and this point is called the **origin** of the real number line. Points to the right of the origin represent positive real numbers, while points to the left of the origin represent negative real numbers.

Figure 2 is an illustration of the real number line with several points plotted. Note that two irrational numbers are plotted, though their locations on the line are necessarily approximations.

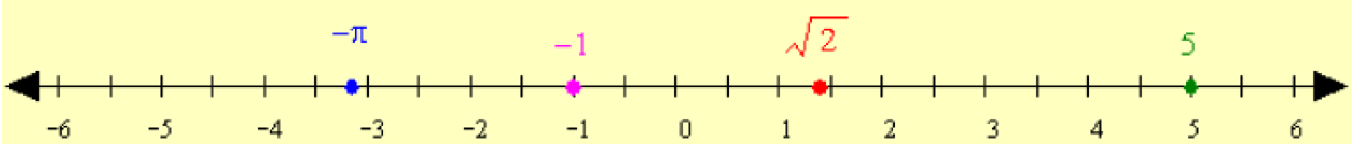


Figure 2: The Real Number Line

Graph inequalities

We choose which portion of the real number line to depict and the physical length that represents one unit as suggested by the numbers that we wish to plot. For example:

Example 2 : Plotting Real Numbers

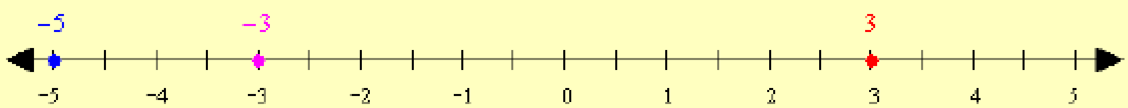
Plot

a. $-5, -3,$ and 3

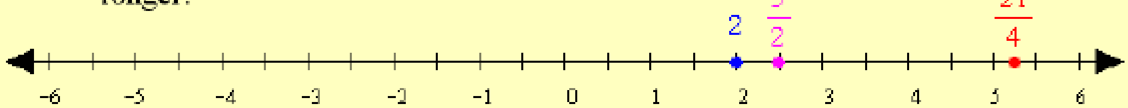
b. $2, \frac{5}{2}$, and $\frac{21}{4}$

Solution:

a. If we want to plot the numbers $-5, -3,$ and 3, we might construct the diagram below.



b. If we want to plot the numbers $2, \frac{5}{2}$, and $\frac{21}{4}$, we might make the unit interval longer.



Inequality Symbols (Order)

Symbol	Meaning
$a < b$ (read "a is less than b")	a lies to the left of b on the number line.
$a \le b$ (read "a is less than or equal to b")	a lies to the left of b or is equal to b.
$b > a$ (read "b is greater than a")	b lies to the right of a on the number line.
$b \ge a$ (read "b is greater than or equal to a")	b lies to the right of a or is equal to a.

The two symbols $<$ and $>$ are called *strict* inequalities, while the symbols $\le$ and $\ge$ are *non–strict* inequalities.

Example 3 : Qualifying Inequalities

a. $5 \le 9$, since 5 lies to the left of 9.

b. $5 \le 5$, since 5 is equal to 5. Note that for every real number a, $a \le a$ and $a \ge a$.

c. $-7 > -163$, since -7 lies to the right of -163.

d. The statement "5 is greater than -2" can be written $5 > -2$.

e. The statement "a is less than or equal to $b + c$" can be written $a \le b + c$.

f. The statement "x is strictly less than y" can be written $x < y$.

g. The negation of the statement $a \le b$ is the statement $a > b$.

h. If $a \le b$ and $a \ge b$, then it must be the case that $a = b$.

Notation	Meaning
(a, b)	$\{x \mid a < x < b\}$, or all real numbers strictly between a and b.
$[a, b]$	$\{x \mid a \le x \le b\}$, or all real numbers between a and b, including both a and b.
$(a, b]$	$\{x \mid a < x \le b\}$, or all real numbers between a and b, including b but not a.
$(-\infty, b)$	$\{x \mid x < b\}$, or all real numbers less than b.
$[a, \infty)$	$\{x \mid x \ge a\}$, or all real numbers greater than or equal to a.

EXAMPLE 4 **Graphing Inequalities**

(a) On the real number line, graph all numbers x for which $x > 4$.
(b) On the real number line, graph all numbers x for which $x \le 5$.

Solution

(a) See Figure 8. Notice that we use a left parenthesis to indicate that the number 4 is *not* part of the graph.

(b) See Figure 9. Notice that we use a right bracket to indicate that the number 5 *is* part of the graph.

Figure 8

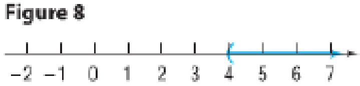
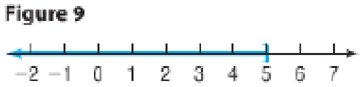


Figure 9



Now Work PROBLEM 41

Sets and their properties

Intersection & Union

DEFINITION

If A and B are sets, the **intersection** of A with B, denoted $A \cap B$, is the set consisting of elements that belong to both A and B. The **union** of A with B, denoted $A \cup B$, is the set consisting of elements that belong to either A or B, or both.

EXAMPLE 2 **Finding the Intersection and Union of Sets**

Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$. Find:

(a) $A \cap B$ (b) $A \cup B$ (c) $B \cap (A \cup C)$

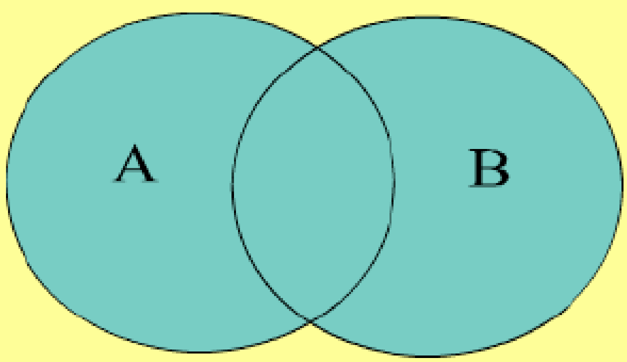
Solution

(a) $A \cap B = \{1, 3, 5, 8\} \cap \{3, 5, 7\} = \{3, 5\}$
(b) $A \cup B = \{1, 3, 5, 8\} \cup \{3, 5, 7\} = \{1, 3, 5, 7, 8\}$
(c) $B \cap (A \cup C) = \{3, 5, 7\} \cap (\{1, 3, 5, 8\} \cup \{2, 4, 6, 8\})$
$= \{3, 5, 7\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{3, 5\}$

Union

In this definition, A and B denote two sets, and are represented in the Venn diagrams by circles. The operations of union and intersection are demonstrated in the diagrams by means of shading.

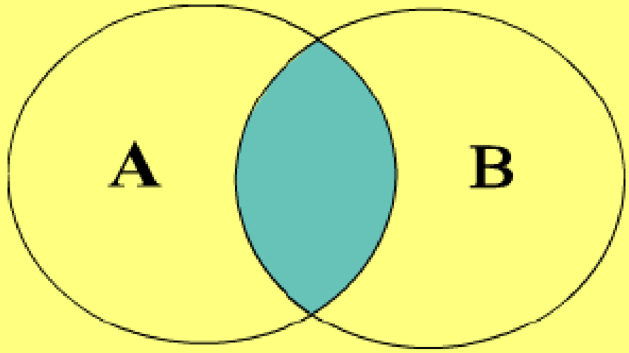
The **union** of A and B, denoted $A \cup B$, is the set $\{ x \mid x \in A \text{ or } x \in B \}$
That is, an element x is in $A \cup B$ if it is in the set A, the set B or both.
Note that the union of A and B contains both individual sets.



Intersection

In this definition, A and B denote two sets, and are represented in the Venn diagrams by circles. The operations of union and intersection are demonstrated in the diagrams by means of shading.

The **intersection** of A and B, denoted $A \cap B$, is the set $\{ x \mid x \in A \text{ and } x \in B \}$ That is, an element x is in $A \cap B$ if it is in both A and B. Note that the intersection of A and B is contained in each individual set.



Simplify each of the following set expressions, if possible

a. $[-5, 5] \cup [1, 7)$

Solution:

Picture these intervals on a number line. Because they overlap, their union can be described as a single interval, from the left-most point to the right-most point.

Answer: $[-5, 7)$

Recall, the bracket is a closed interval, meaning that the point is included in the interval. The parenthesis is an open interval, meaning the point itself is not included.

b. $[-5, 5] \cap [1, 7)$

Solution:

We already realized in part a. that these intervals overlap. Their intersection consists of those parts contained in both, or the overlapping section.

Answer: $[1, 5]$

Be sure to carry forward the same notation. Do not replace a closed interval with an open one, or vice versa.

c. $(-2, 6) \cap (6, 10]$

Solution:

Because these intervals are open at the point 6, they do not overlap. They have no elements in common, and their intersection is the ∅.

Answer: ∅

d. $(-\infty, 10) \cup [7, \infty)$

Solution:

These intervals do overlap. Thus, their union ranges from negative infinity to infinity, and represents the entire set of real numbers.

Answer: $(-\infty, \infty)$

Absolute Value

Absolute Value and Distance on the Real Number Line

In addition to order, the depiction of the set of real numbers as a line leads to the notion of *distance*. Physically, distance is a well-understood concept: the distance between two objects is a non-negative number, dependent on a choice of measuring system, indicating how close the objects are to one another. The mathematical idea of **absolute value** gives us a means of defining distance in a mathematical setting.

Absolute Value

The **absolute value** of a real number a, denoted as $|a|$, is defined by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

The absolute value of a number is also referred to as its **magnitude**; it is the non-negative number corresponding to its distance from the origin.

Note that this definition implicitly gives us a system of measurement: 1 and -1 are the two real numbers which have a magnitude of 1, and so the distance between 0 and 1 (or between -1 and 0) is one unit. Note also that 0 is the only real number whose absolute value is 0.

a. $|-\pi| = |\pi| = \pi$. Both $-\pi$ and π are π units from 0.

b. $|17-3| = |3-17| = 14$. 17 and 3 are 14 units apart.

c. $\dfrac{|-7|}{-7} = \dfrac{7}{-7} = -1$.

d. $\dfrac{|7|}{7} = \dfrac{7}{7} = 1$.

e. $-|-5| = -5$. **Note:** The negative sign outside the absolute value symbol is not affected by the absolute value. Compare this with the fact that $-(-5) = 5$.

f. $|\sqrt{7}-2| = \sqrt{7}-2$. Even without a calculator, we know $\sqrt{7}$ is larger than 2 (since $2 = \sqrt{4}$), so $\sqrt{7}-2$ is positive and hence
$|\sqrt{7}-2| = \sqrt{7}-2$.

g. $|\sqrt{7}-19| = 19-\sqrt{7}$. In contrast to the last example, we know $\sqrt{7}-19$ is negative, so its absolute value is $19-\sqrt{7}$.

The previous examples illustrate some of the basic properties of absolute value. The list of properties below can all be derived from the definition of absolute value.

Properties of Absolute Values

For all real numbers a and b,

1. $|a| \geq 0$

2. $|-a| = |a|$

3. $a \leq |a|$

4. $|ab| = |a||b|$

5. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

6. $|a+b| \leq |a| + |b|$. (This is called the **triangle inequality**, as it is a reflection of the fact that one side of a triangle is never longer than the sum of the other two sides.)

The following example also illustrates some of the properties of absolute values.

a. $|(-3)(5)| = |-15| = 15 = |-3||5|$.

b. $1 = |-3+4| \leq |-3| + |4| = 7$.

c. $7 = |-3-4| \leq |-3| + |-4| = 7$.

d. $\left|\frac{-3}{7}\right| = \frac{|-3|}{|7|} = \frac{3}{7}$.

Properties of Exponents

Throughout this table, a and b may be taken to represent constants, variables, or more complicated algebraic expressions. The letters n and m represent integers.

	Property	Example
1.	$a^n \cdot a^m = a^{n+m}$	$(-3)^3 \cdot (-3)^{-1} = (-3)^{3+(-1)} = (-3)^2 = 9$
2.	$\dfrac{a^n}{a^m} = a^{n-m}$	$\dfrac{7^9}{7^{10}} = 7^{9-10} = 7^{-1}$
3.	$a^{-n} = \dfrac{1}{a^n}$	$5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$ and $x^3 = \dfrac{1}{x^{-3}}$
4.	$(a^n)^m = a^{nm}$	$(2^3)^2 = 2^{3\cdot 2} = 2^6 = 64$

Properties of Exponents, cont.

	Property	Example
5.	$(ab)^n = a^n b^n$	$(7x)^3 = 7^3x^3 = 343x^3$ and $(-2x^5)^2 = (-2)^2(x^5)^2 = 4x^{10}$
6.	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{3}{x}\right)^2 = \dfrac{3^2}{x^2} = \dfrac{9}{x^2}$ and $\left(\dfrac{1}{3z}\right)^2 = \dfrac{1^2}{(3z)^2} = \dfrac{1}{9z^2}$
7.	$\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$	$\left(\dfrac{5}{4}\right)^{-3} = \dfrac{4^3}{5^3} = \dfrac{64}{125}$
8.	$\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$	$\dfrac{3^{-2}}{2^{-4}} = \dfrac{2^4}{3^2} = \dfrac{16}{9}$

In the above table, it is assumed that every expression is defined. That is, if an exponent is 0, then the base is non-zero, and if an expression appears in the denominator of a fraction, then that expression is non-zero. Remember that $a^0 = 1$ for every $a \neq 0$.

Incorrect Statements	Corrected Statements
$x^3 \cdot x^6 = x^{18}$	$x^3 \cdot x^6 = x^9$
$2^5 \cdot 2^4 = 4^9$	$2^5 \cdot 2^4 = 2^9$
$(x^3 + 6y)^{-1} = \frac{1}{x^3} + \frac{1}{6y}$	$(x^3 + 6y)^{-1} = \frac{1}{x^3 + 6y}$
$(5x)^2 = 5x^2$	$(5x)^2 = 25x^2$

Polynomials

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is the **leading coefficient,** and n is the **degree** of the polynomial.

The monomials that make up a polynomial are called its **terms.** If all the coefficients are 0, the polynomial is called the **zero polynomial,** which has no degree.

Polynomials are usually written in **standard form,** beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree. If a power of x is missing, it is because its coefficient is zero.

Field Properties

In this table, a, b, and c represent arbitrary real numbers. The first five properties apply to addition and multiplication, while the last combines the two.

Name of Property	Additive Version	Multiplicative Version
Closure	$a + b$ is a real number	ab is a real number
Commutative	$a + b = b + a$	$ab = ba$
Associative	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Identity	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Inverse	$a + (-a) = 0$	$a \frac{1}{a} = 1$ (for $a \neq 0$)
Distributive	$a(b + c) = ab + ac$	

Examples of Polynomials

Polynomial	Coefficients	Degree
$-8x^3 + 4x^2 + 6x + 2$	$-8, 4, 6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x + (-5)$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 + (-2)x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree

a. Consider the following expression:

$$-27y^5(2x^3 + y) + 3\sqrt{y} - 7(x + y^2)$$

The following table illustrates the elements of this expression and their names.

Algebraic Expressions	$-27y^5(2x^3 + y) + 3\sqrt{y} - 7(x + y^2)$		
Terms	$-27\,y^5(2x^3 + y)$	$3\sqrt{y}$	$-7(x + y^2)$
Factors	$-27,\ y^5,\ (2x^3 + y)$	$3,\ \sqrt{y}$	$-7,\ (x + y^2)$
Coefficient (of term)	-27	3	-7
Variable Factor (of term)	$y^5(2x^3 + y)$	$\sqrt{y}$	$(x + y^2)$

Note that -27, 3, and -7 are simultaneously **factors** and **coefficients**. Also, both $(2x^3 + y)$ and $(x + y^2)$ are made up of two *terms*, but they are not terms of the *original expression*, just terms of those *factors*.

Order of Operations

1. If the expression is a fraction, simplify the numerator and denominator individually, according to the guidelines in the following steps.
2. **Parentheses, braces, and brackets** are all used as grouping symbols. Simplify expressions within each set of grouping symbols, if they are present, working from the innermost outward.
3. Simplify all **powers (exponents) and roots**.
4. Perform all **multiplications and divisions** in the expression in the order they occur, working from left to right.
5. Perform all **additions and subtractions** in the expression in the order they occur, working from left to right.