The Rational Zero Theorem

The Rational Zero Theorem gives a list of **possible** rational zeros of a polynomial function. Equivalently, the theorem gives all possible rational roots of a polynomial equation. Not every number in the list will be a zero of the function, but every rational zero of the polynomial function will appear somewhere in the list.

The Rational Zero Theorem

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients and $\frac{p}{q}$ (where $\frac{p}{q}$ is reduced) is a rational zero, then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

EXAMPLE: Using the Rational Zero Theorem

List all possible rational zeros of $f(x) = 15x^3 + 14x^2 - 3x - 2$.

Solution The constant term is –2 and the leading coefficient is 15.

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term, } -2}{\text{Factors of the leading coefficient, } 15}$$

$$= \frac{\pm 1, \pm 2}{\pm 1, \pm 3, \pm 5, \pm 15}$$

$$= \pm 1, \quad \pm 2, \qquad \pm \frac{1}{3}, \quad \pm \frac{2}{3}, \qquad \pm \frac{1}{5}, \quad \pm \frac{2}{5}, \qquad \pm \frac{1}{15}, \quad \pm \frac{2}{15}$$

Divide ±1 and ±2 by ±1.	**Divide ±1 and ±2 by ±3.**	**Divide ±1 and ±2 by ±5.**	**Divide ±1 and ±2 by ±15.**

There are 16 possible rational zeros. The actual solution set to $f(x) = 15x^3 + 14x^2 - 3x - 2 = 0$ is $\{-1, -\frac{1}{3}, \frac{2}{5}\}$, which contains 3 of the 16 possible solutions.

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0$.

Solution Because we are given an equation, we will use the word "**roots**," rather than "zeros," in the solution process. We begin by listing all possible rational roots.

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term, } 24}{\text{Factors of the leading coefficient, } 1}$$

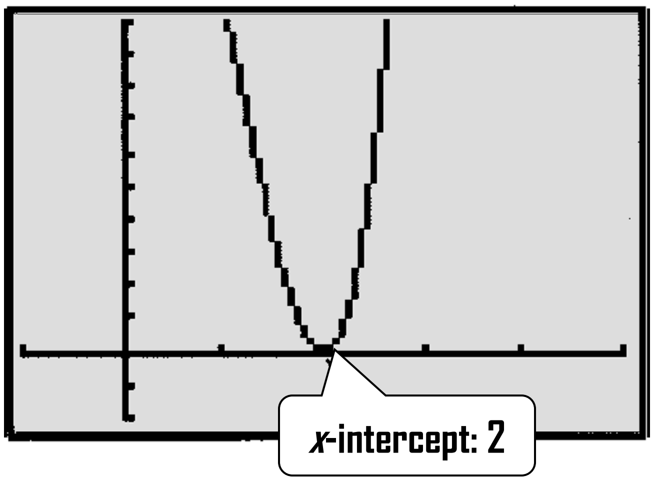
$$= \frac{\pm 1, \pm 2 \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24}{\pm 1}$$

$$= \pm 1, \pm 2 \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$$

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0.$

Solution The graph of $f(x) = x^4 - 6x^2 - 8x + 24$ is shown the figure below. Because the x-intercept is 2, we will test 2 by synthetic division and show that it is a root of the given equation.



$$\begin{array}{r|rrrrr} 2 & 1 & 0 & -6 & -8 & 24 \\ & & 2 & 4 & -4 & -24 \\ \hline & 1 & 2 & -2 & -12 & 0 \end{array}$$

The zero remainder indicates that 2 is a root of $x^4 - 6x^2 - 8x + 24 = 0$.

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0.$

Solution Now we can rewrite the given equation in factored form.

$$x^4 - 6x^2 + 8x + 24 = 0$$ **This is the given equation.**

$$(x - 2)(x^3 + 2x^2 - 2x - 12) = 0$$ **This is the result obtained from the synthetic division.**

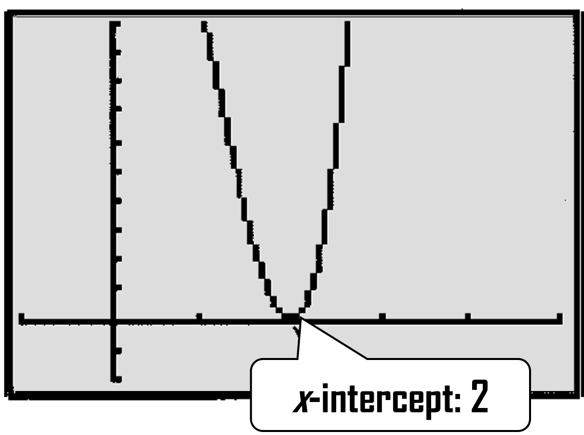
$$x - 2 = 0 \quad \text{or} \quad x^3 + 2x^2 - 2x - 12 = 0$$ **Set each factor equal to zero.**

Now we must continue by factoring $x^3 + 2x^2 - 2x - 12 = 0$

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0.$

Solution Because the graph turns around at 2, this means that 2 is a root of even multiplicity. Thus, 2 must also be a root of $x^3 + 2x^2 - 2x - 12 = 0.$



These are the coefficients of $x^3 + 2x^2 - 2x - 12 = 0.$

$$
\begin{array}{r|rrrr}
\underline{2} & 1 & 2 & -2 & -12 \\
 & & 2 & 8 & 12 \\
\hline
 & 1 & 4 & 6 & 0
\end{array}
$$

The zero remainder indicates that 2 is a root of $x^3 + 2x^2 - 2x - 12 = 0.$

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0.$

Solution Now we can solve the original equation as follows.

$$x^4 - 6x^2 + 8x + 24 = 0$$

This is the given equation.

$$(x - 2)(x^3 + 2x^2 - 2x - 12) = 0$$

This was obtained from the first synthetic division.

$$(x - 2)(x - 2)(x^2 + 4x + 6) = 0$$

This was obtained from the second synthetic division.

$$x - 2 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x^2 + 4x + 6 = 0$$

Set each factor equal to zero.

$$x = 2 \qquad x = 2 \qquad x^2 + 4x + 6 = 0$$

Solve.

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0.$

Solution We can use the quadratic formula to solve $x^2 + 4x + 6 = 0.$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We use the quadratic formula because $x^2 + 4x + 6 = 0$ cannot be factored.

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)}$$

Let $a = 1$, $b = 4$, and $c = 6$.

$$= \frac{-4 \pm \sqrt{-8}}{2}$$

Multiply and subtract under the radical.

$$= \frac{-4 \pm 2i\sqrt{2}}{2}$$

$\sqrt{-8} = \sqrt{4(2)(-1)} = 2i\sqrt{2}$

$$= -2 \pm i\sqrt{2}$$

Simplify.

The solution set of the original equation is $\{2, -2 - i\sqrt{2}, -2 + i\sqrt{2}\}$.

Properties of Polynomial Equations

1. If a polynomial equation is of degree n, then counting multiple roots separately, the equation has n roots.

2. If $a + bi$ is a root of a polynomial equation ($b \neq 0$), then the non-real complex number $a - bi$ is also a root. Non-real complex roots, if they exist, occur in conjugate pairs.

For each of the polynomials that follow, list all of the potential rational zeros. Then write the polynomial in factored form and identify the actual zeros.

a. $f(x) = 3x^3 - 8x^2 - 20x + 16$

$f(x) = 3x^3 - 8x^2 - 20x + 16$

Factors of a_0 : $\pm\{1, 2, 4, 8, 16\}$

Factors of a_3 : $\pm\{1, 3\}$

Possible rational zeros:

$$\pm\left\{ \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, 2, \frac{8}{3}, 4, \frac{16}{3}, 8, 16 \right\}$$

We begin by listing the factors of the constant term, 16, and the leading coefficient, 3. Note that each number has both positive and negative factors.

By the Rational Zero Theorem, any rational zero must be one of the twenty numbers generated by dividing factors of a_0 by factors of a_3.

$$\begin{array}{r|rrrr} -2 & 3 & -8 & -20 & 16 \\ & & -6 & 28 & -16 \\ \hline & 3 & -14 & 8 & 0 \end{array}$$

We can then use synthetic division to identify which of the potential zeros actually *are* zeros. Here, we first discover that –2 is a zero.

$$\begin{aligned} f(x) &= (x+2)(3x^2 - 14x + 8) \\ &= (x+2)(3x-2)(x-4) \end{aligned}$$

Then finish by factoring the trinomial $3x^2 - 14x + 8$. Often, you may have to try several potential zeros before finding an actual zero.

Actual zeros: $\left\{ -2, \frac{2}{3}, 4 \right\}$