Objective 09 Understand, solve, graph and apply exponential and logarithmic equations including familiarity with the change of base formula to evaluate logarithms

Exponentials equations

Definition of the Exponential Function

> The exponential function f with base b is defined by
>
> $$f(x) = b^x \quad \text{or} \quad y = b^x$$
>
> Where b is a positive constant other than and x is any real number.

Here are some examples of exponential functions.

$f(x) = 2^x$ — Base is 2.

$g(x) = 10^x$ — Base is 10.

$h(x) = 3^{x+1}$ — Base is 3.

Understand & Solve

The value of $f(x) = 3^x$ when $x = 2$ is

$$f(2) = 3^2 = 9$$

The value of $f(x) = 3^x$ when $x = -2$ is

$$f(-2) = 3^{-2} = \frac{1}{9}$$

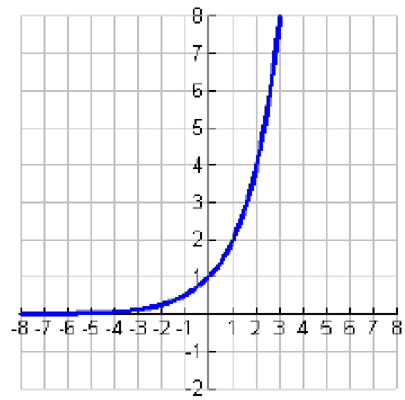
The value of $g(x) = 0.5^x$ when $x = 4$ is

$$g(4) = 0.5^4 = 0.0625$$

Graph

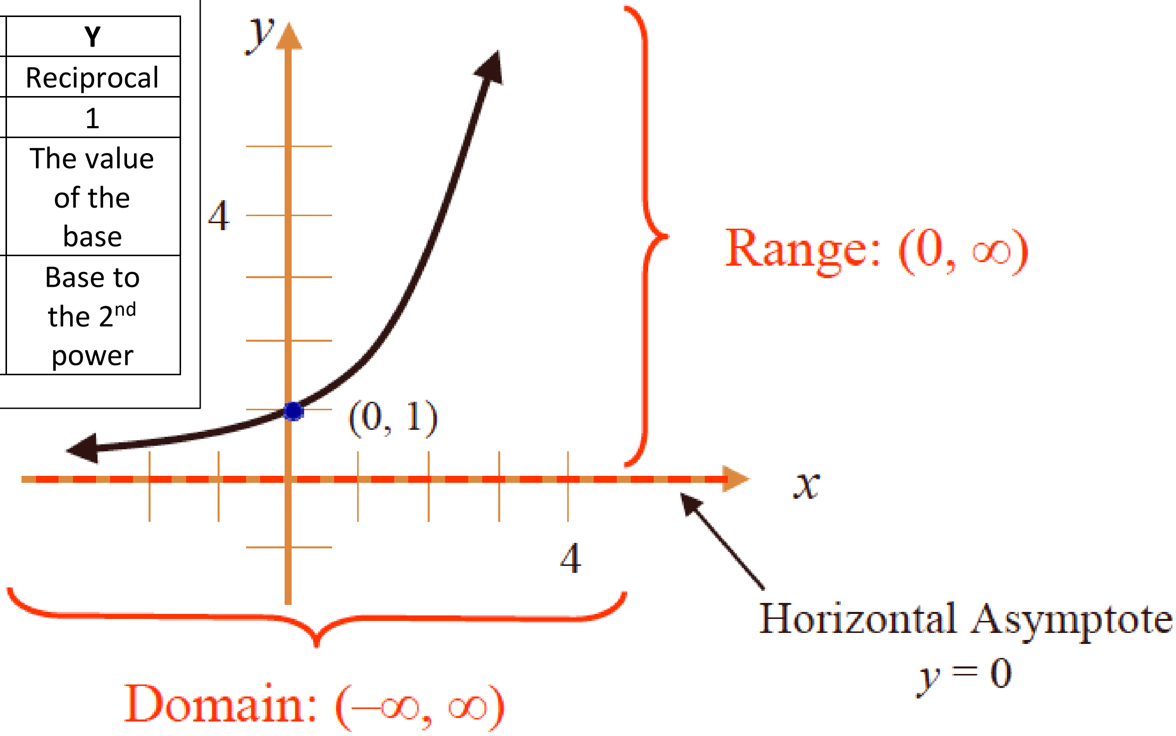
Definition of Exponential Functions

- The exponential function *f* with a base *b* is defined by $f(x) = b^x$ where b is a positive constant other than 1 ($b > 0$, and $b \neq 1$) and x is any real number.
- So, $f(x) = 2^x$, looks like:

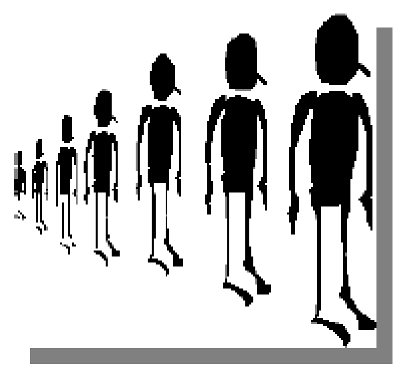


The graph of $f(x) = a^x$, $a > 1$

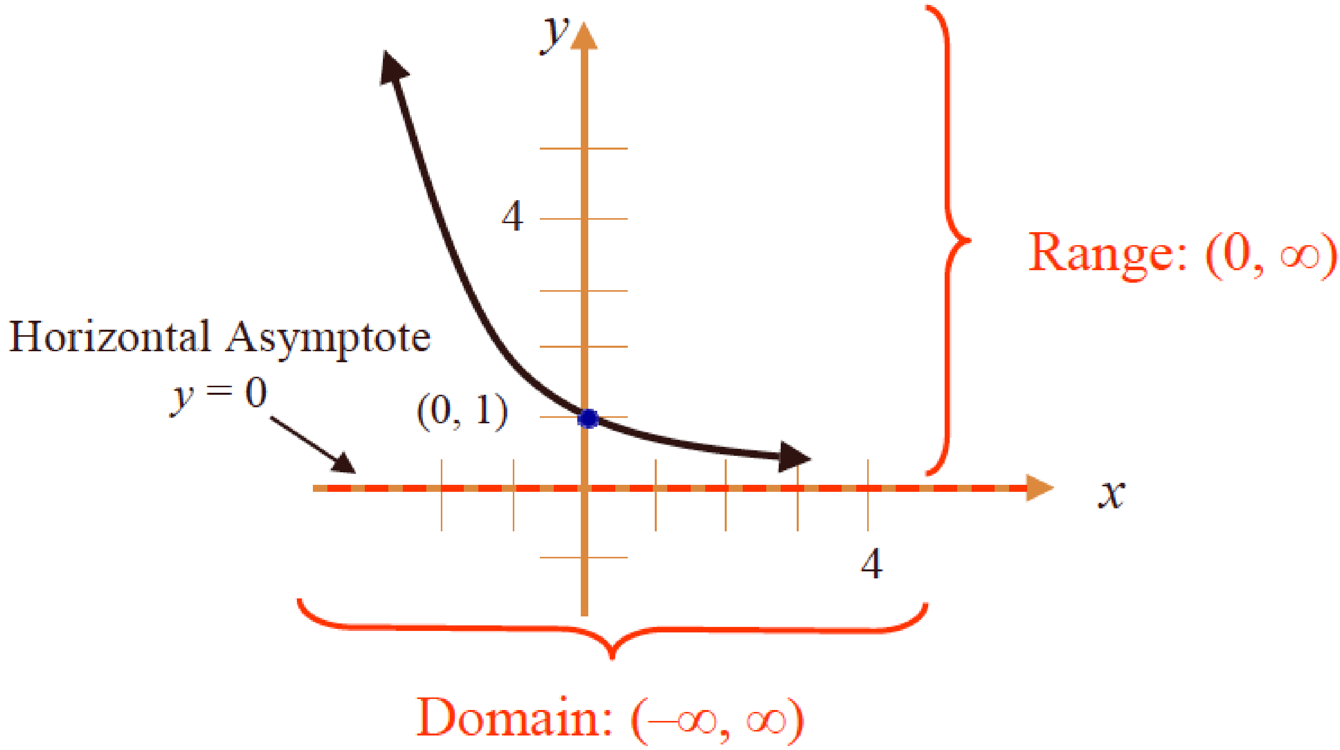
X	Y
-1	Reciprocal
0	1
1	The value of the base
2	Base to the 2nd power



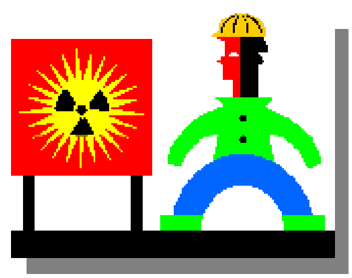
- Population growth often modeled by exponential function



The graph of $f(x) = a^x$, $0 < a < 1$



- Half life of radioactive materials modeled by exponential function



Decreasing Exponentials

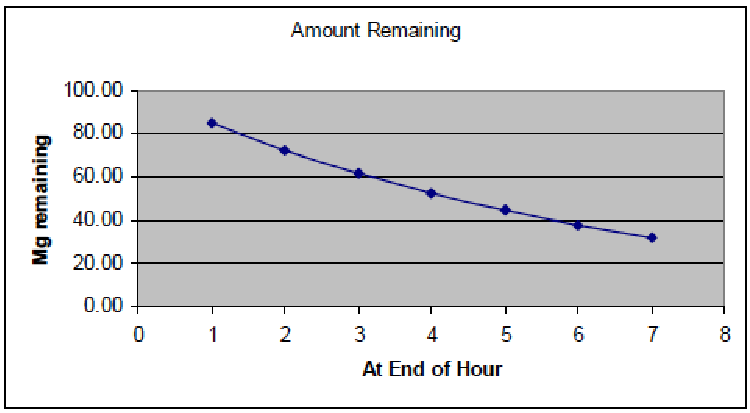
- Completed chart

Growth Factor = 0.85

Note: when growth factor < 1, exponential is a decreasing function

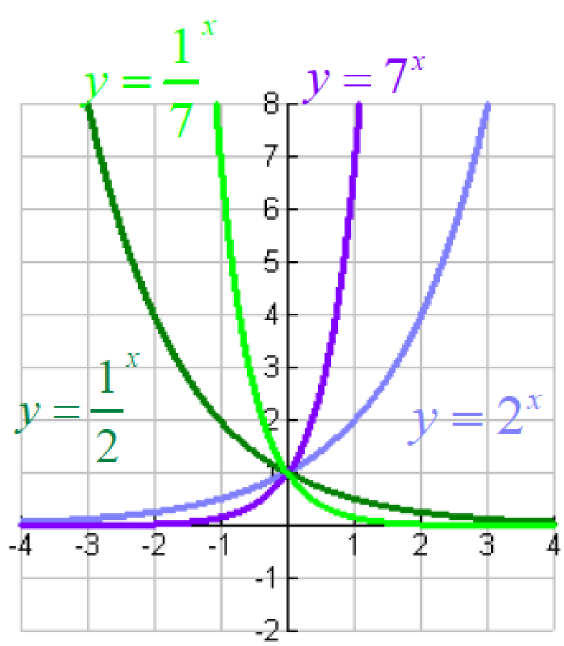
At end of hour	Amount Remaining
1	85.00
2	72.25
3	61.41
4	52.20
5	44.37
6	37.71
7	32.06

- Graph



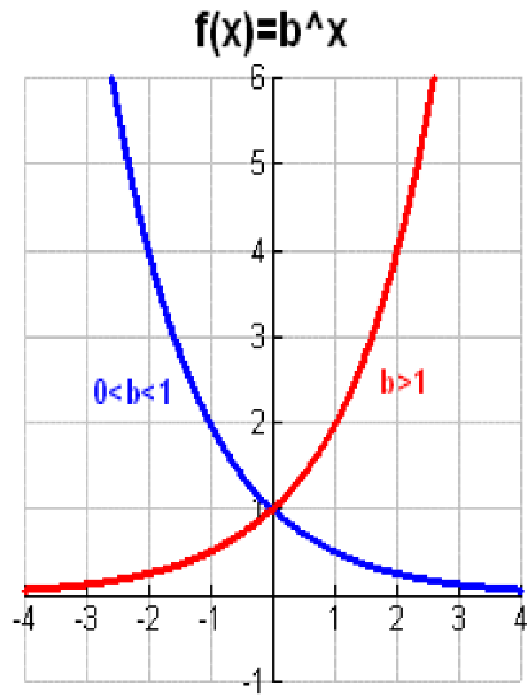
Graphing Exponential Functions

- Four exponential functions have been graphed. Compare the graphs of functions where b > 1 to those where b < 1



Graphing Exponential Functions

- So, when b > 1, f(x) has a graph that goes up to the right and is an increasing function.
- When 0 < b < 1, f(x) has a graph that goes down to the right and is a decreasing function.



Characteristics

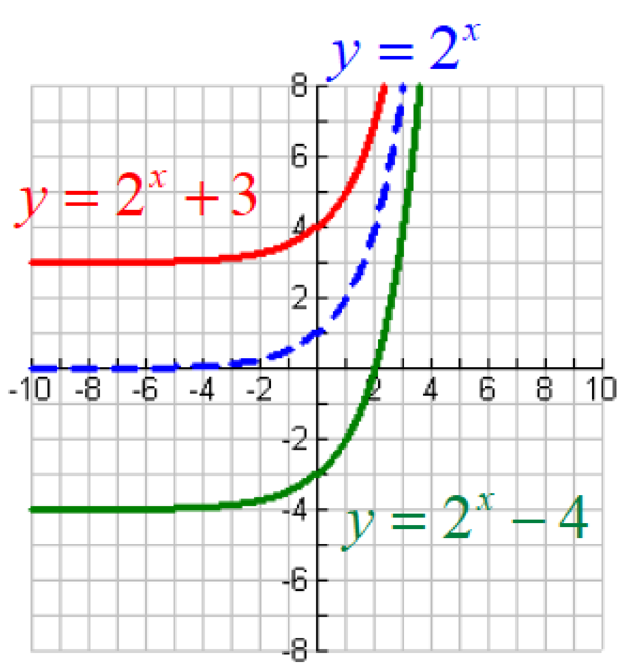
- The domain of $f(x) = b^x$ consists of all real numbers $(-\infty, \infty)$. The range of $f(x) = b^x$ consists of all positive real numbers $(0, \infty)$.
- The graphs of all exponential functions pass through the point (0,1). This is because $f(o) = b^0 = 1$ $(b \neq o)$.
- The graph of $f(x) = b^x$ approaches but does not cross the x-axis. The x-axis is a horizontal asymptote.
- $f(x) = b^x$ is one-to-one and has an inverse that is a function.

Transformations Defined

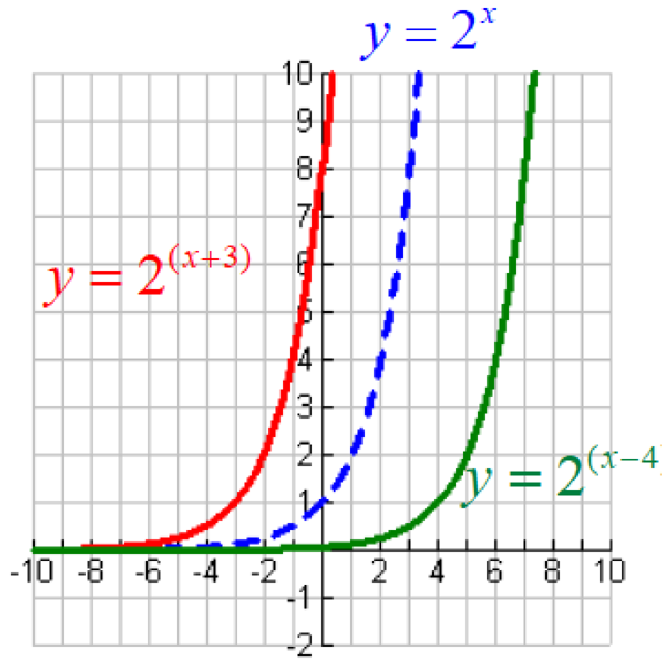
Transformation	Equation	Description
Horizontal translation	$g(x) = b^{x+c}$	• Shifts the graph of $f(x) = b^x$ to the left c units if $c > 0$. • Shifts the graph of $f(x) = b^x$ to the right c units if $c < 0$.
Vertical stretching or shrinking	$g(x) = c\,b^x$	Multiplying y-coordintates of $f(x) = b^x$ by c, • Stretches the graph of $f(x) = b^x$ if $c > 1$. • Shrinks the graph of $f(x) = b^x$ if $0 < c < 1$.
Reflecting	$g(x) = -b^x$ $g(x) = b^{-x}$	• Reflects the graph of $f(x) = b^x$ about the x-axis. • Reflects the graph of $f(x) = b^x$ about the y-axis.
Vertical translation	$g(x) = -b^x + c$	• Shifts the graph of $f(x) = b^x$ upward c units if $c > 0$. • Shifts the graph of $f(x) = b^x$ downward c units if $c < 0$.

Transformations

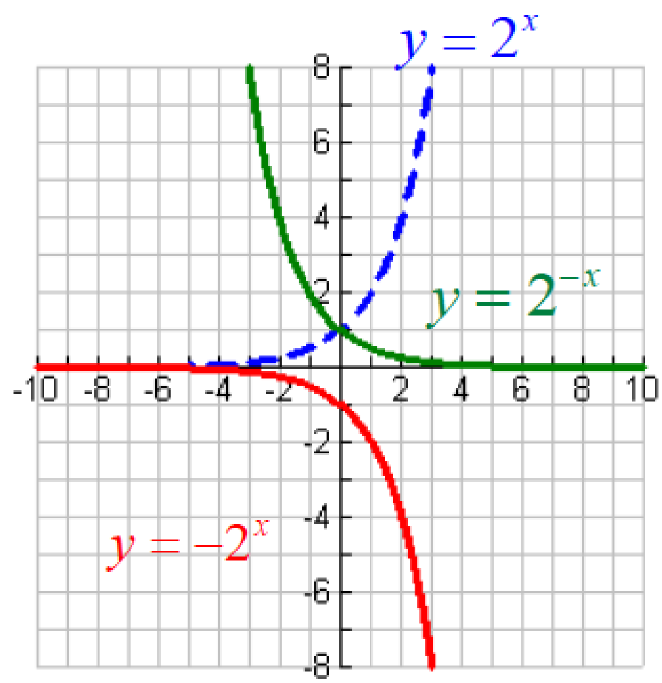
- **Vertical translation $f(x) = b^x + c$**
- Shifts the graph up if $c > 0$
- Shifts the graph down if $c < 0$



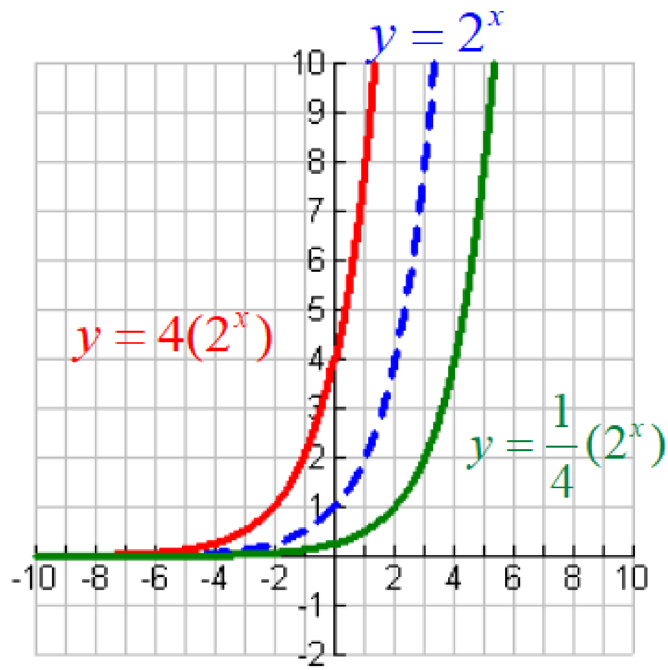
- **Horizontal translation: $g(x)=b^{x+c}$**
- Shifts the graph to the left if c > 0
- Shifts the graph to the right if c < 0



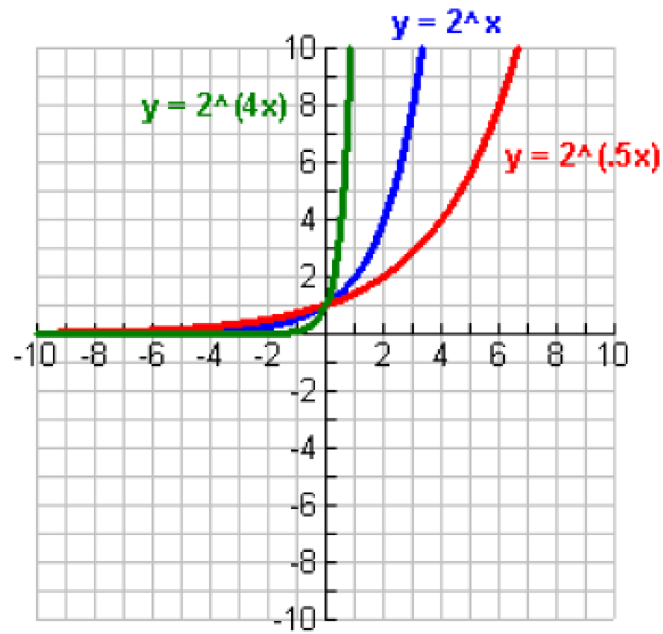
- **Reflecting**
- $g(x) = -b^x$ reflects the graph about the x-axis.
- $g(x) = b^{-x}$ reflects the graph about the y-axis.



- **Vertical stretching or shrinking, $f(x)=cb^x$:**
- Stretches the graph if c > 1
- Shrinks the graph if 0 < c < 1



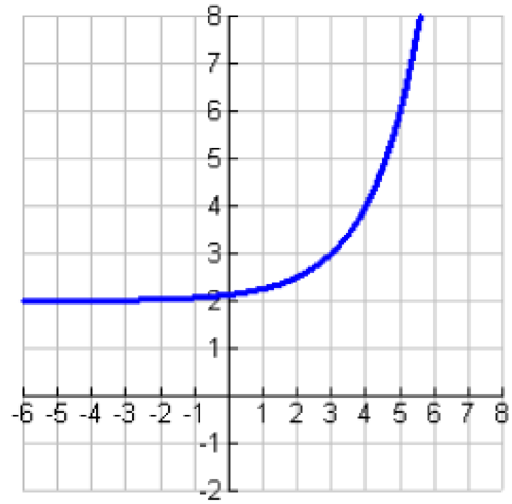
- **Horizontal stretching or shrinking, $f(x)=b^{cx}$:**
- Shinks the graph if c > 1
- Stretches the graph if 0 < c < 1



EXCERCISES

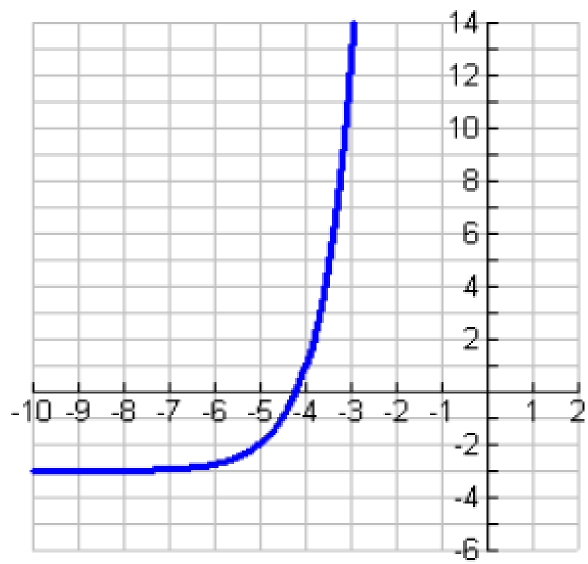
- Graph the function $f(x) = 2^{(x-3)} + 2$
- Where is the horizontal asymptote?

$y = 2$



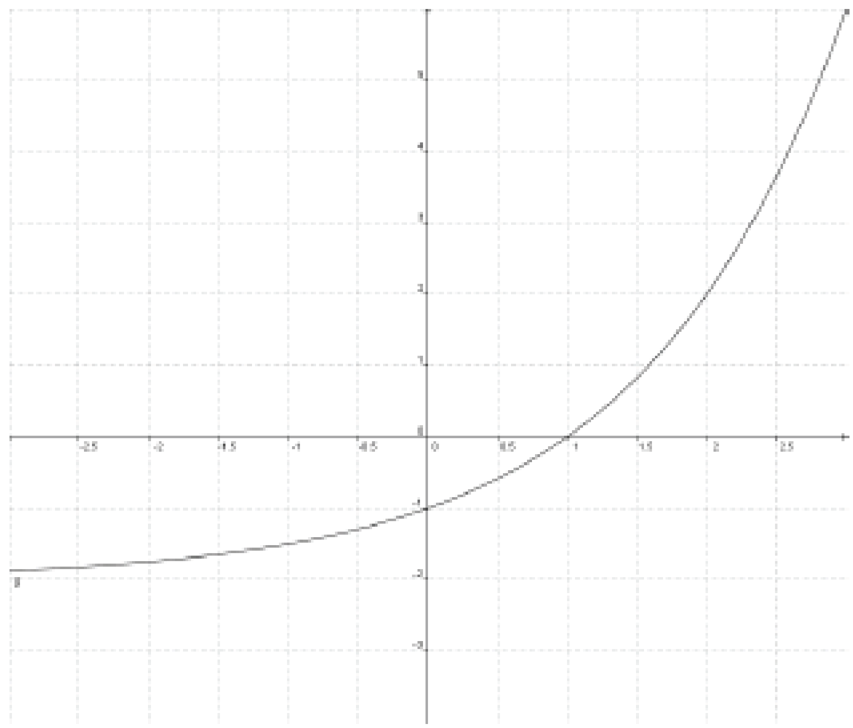
- Graph the function $f(x) = 4^{(x+5)} - 3$
- Where is the horizontal asymptote?

$y = -3$



Which function matches the graph shown in the following graph ?

(a) $y = 2^{x+2}$ (b) $y = 2^{x+1} + 2$ (c) $y = 2^{x-2}$ (d) $y = 2^{x} - 2$

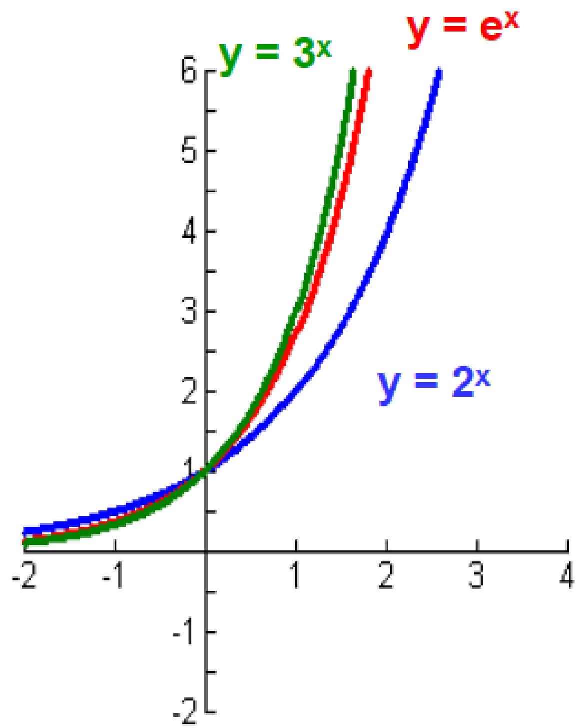


The Number e

- The number e is known as Euler's number. Leonard Euler (1700's) discovered it's importance.
- The number *e* has physical meaning. It occurs naturally in any situation where a quantity increases at a rate proportional to its value, such as a bank account producing interest, or a population increasing as its members reproduce.

The Number e - Definition

- Since $2 < e < 3$, the graph of $y = e^x$ is between the graphs of $y = 2^x$ and $y = 3^x$



- e^x is the 2nd function on the ln key on your calculator

Natural Base

- The irrational number e, is called the natural base.
- The function $f(x) = e^x$ is called the natural exponential function.

Apply equations

Solve the following equations. If there is no solution, state "No Solution".

$\left(\frac{1}{3}\right)^{3x+5} = 9^x$	$5^{-x-9} = 625$	$\left(\frac{1}{2}\right)^{5x+5} = \left(\frac{1}{4}\right)^4$	$2^{x^2+5x} = 4^{-3}$

Logarithmic equations

$$\log_a x = y \leftrightarrow x = a^y$$

A logarithm is an exponent!

- Remember: Logarithmic functions are inverses of exponential functions.

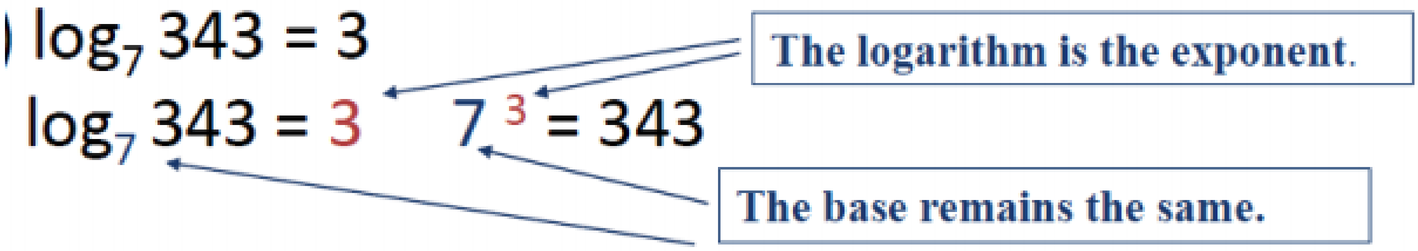
The inverse of $f(x) = a^x$

is given by

$f^{-1}(x) = \log_a x$

Understand & Solve,

$y = \log_b x$ Logarithmic Equation	$x = b^y$ Equivalent Exponential Equation	Solution
$y = \log_2 16$	$16 = 2^y$	$16 = 2^4 \rightarrow y = 4$
$y = \log_2(\frac{1}{2})$	$\frac{1}{2} = 2^y$	$\frac{1}{2} = 2^{-1} \rightarrow y = -1$
$y = \log_4 16$	$16 = 4^y$	$16 = 4^2 \rightarrow y = 2$
$y = \log_5 1$	$1 = 5^y$	$1 = 5^0 \rightarrow y = 0$



Graph

Facts about the Graph of a Logarithmic Function $f(x) = \log_b x$

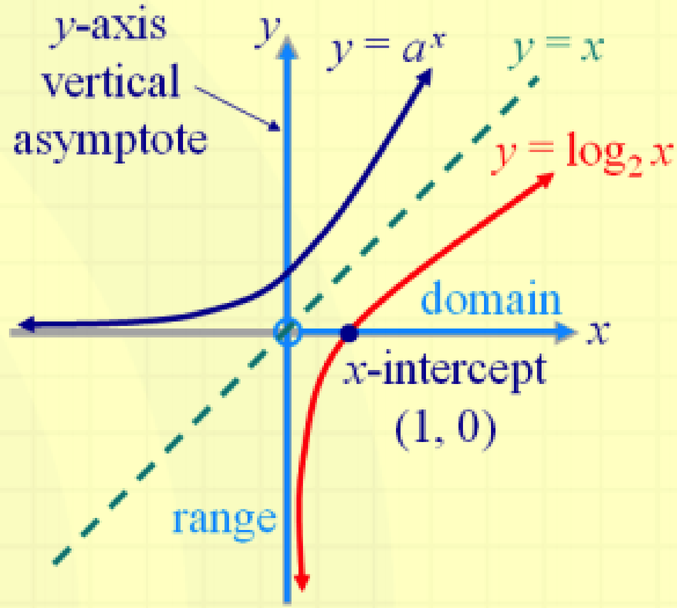
1. The x-intercept of the graph is 1. There is no y-intercept.
2. The y-axis is a vertical asymptote of the graph.
3. A logarithmic function is decreasing if $0 < b < 1$ and increasing if $b > 1$.
4. The graph is smooth and continuous, with no corners or gaps.

The graphs of logarithmic functions are similar for different values of a.

$$f(x) = \log_a x \quad (a > 1)$$

Graph of $f(x) = \log_a x \ (a > 1)$

1. domain $(0, \infty)$
2. range $(-\infty, +\infty)$
3. x-intercept $(1, 0)$
4. vertical asymptote $x = 0$ as $x \to 0^+ \ f(x) \to -\infty$
5. increasing
6. continuous
7. one-to-one
8. reflection of $y = a^x$ in $y = x$



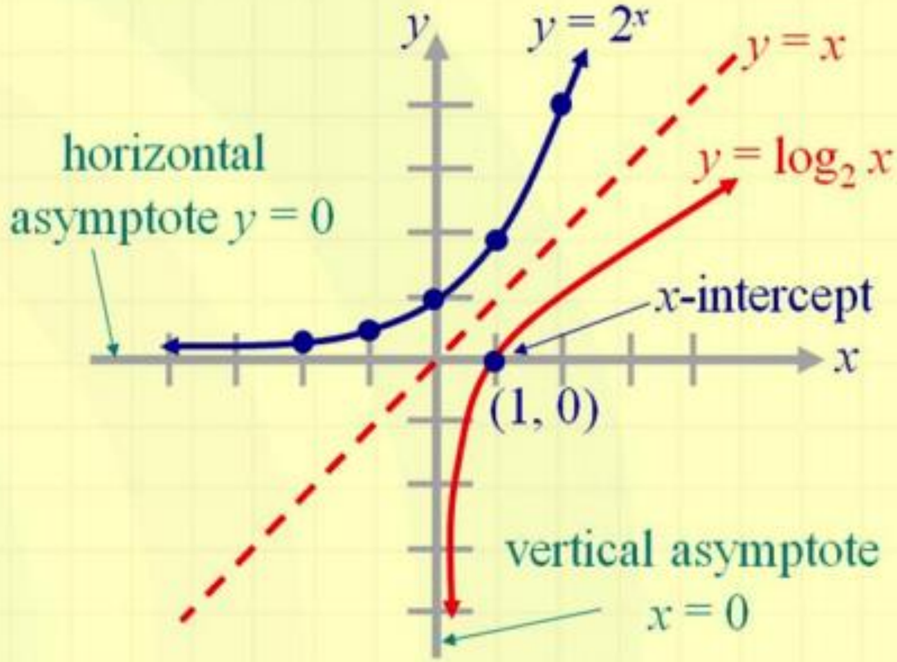
$y = \log_b x$ has the following properties

- Domain $(0, \infty)$, Range $(-\infty, \infty)$
- It passes through the point (1,0)
- It passes through the point (b, 1)
- The y- axis is an asymptote.
- If $b > 1$, it is an increasing function
- If $0 < b < 1$, it is a decreasing function

Graph $f(x) = \log_2 x$

Since the logarithm function is the *inverse* of the exponential function of the same base, its graph is the reflection of the exponential function in the line $y = x$.

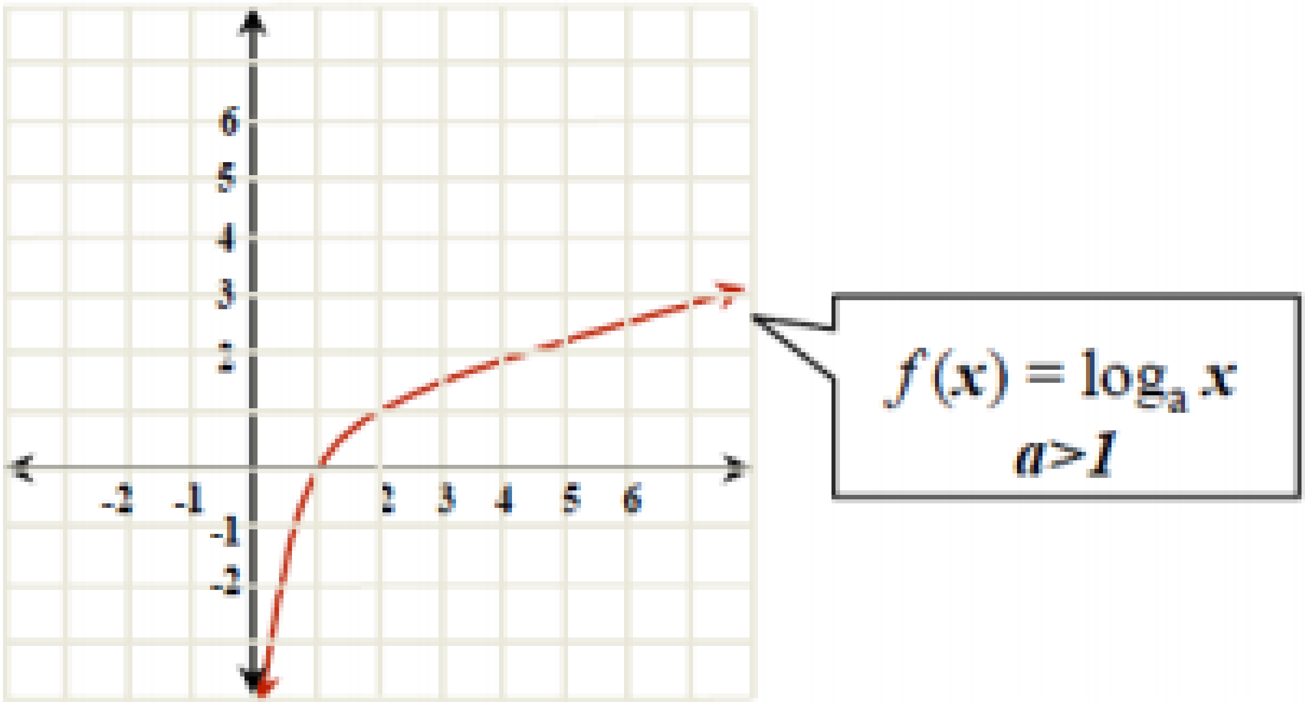
x	2^x
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Transformations Involving Logarithmic Functions

Vertical Translation	$f(x) = c + \log_b x$ $f(x) = -c + \log_b x$	Up c units Down c units
Horizontal Translation	$f(x) = \log_b (x + c)$ $f(x) = \log_b (x - c)$	Left c units Right c units
Stretching: Vertical Horizontal	 $f(x) = c \log_b x$ $f(x) = \log_b cx$	 Stretches by c Stretches by 1/c
Reflection	$f(x) = \log_b (-x)$ $f(x) = -\log_b x$	About y-axis About x-axis

Apply equations



LOGARITHMIC FUNCTION $\quad f(x) = \log_a x, \quad a > 1$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

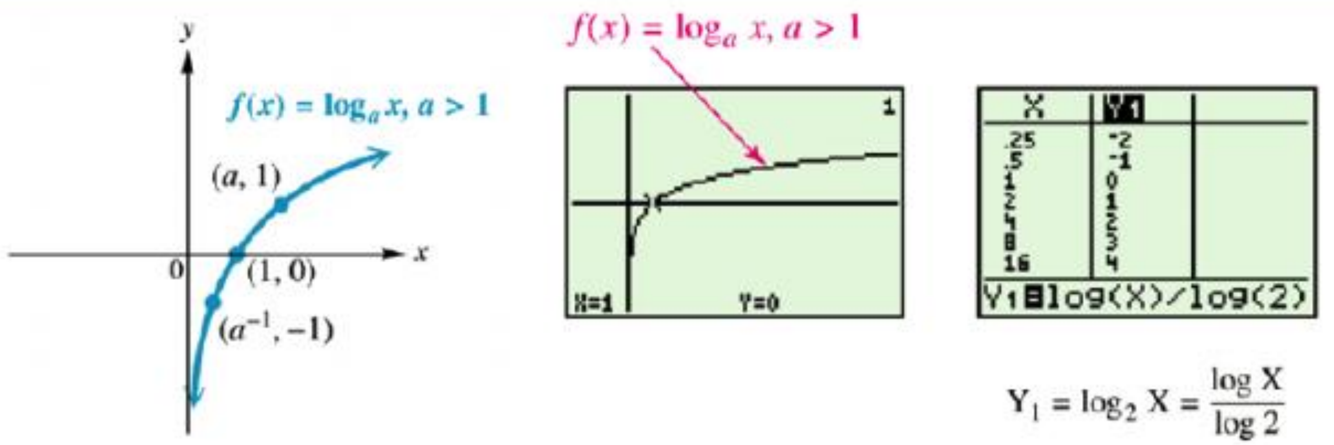
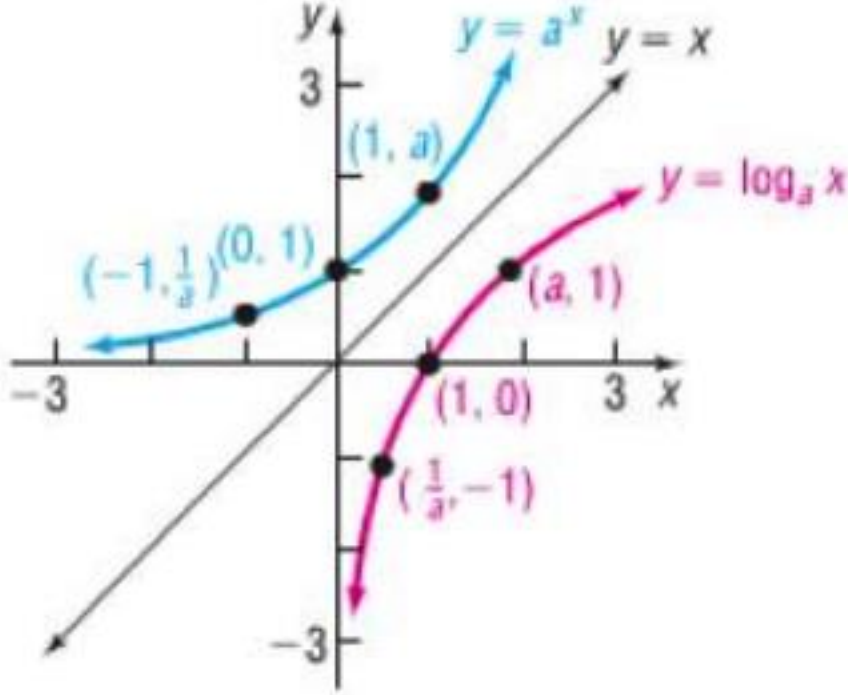
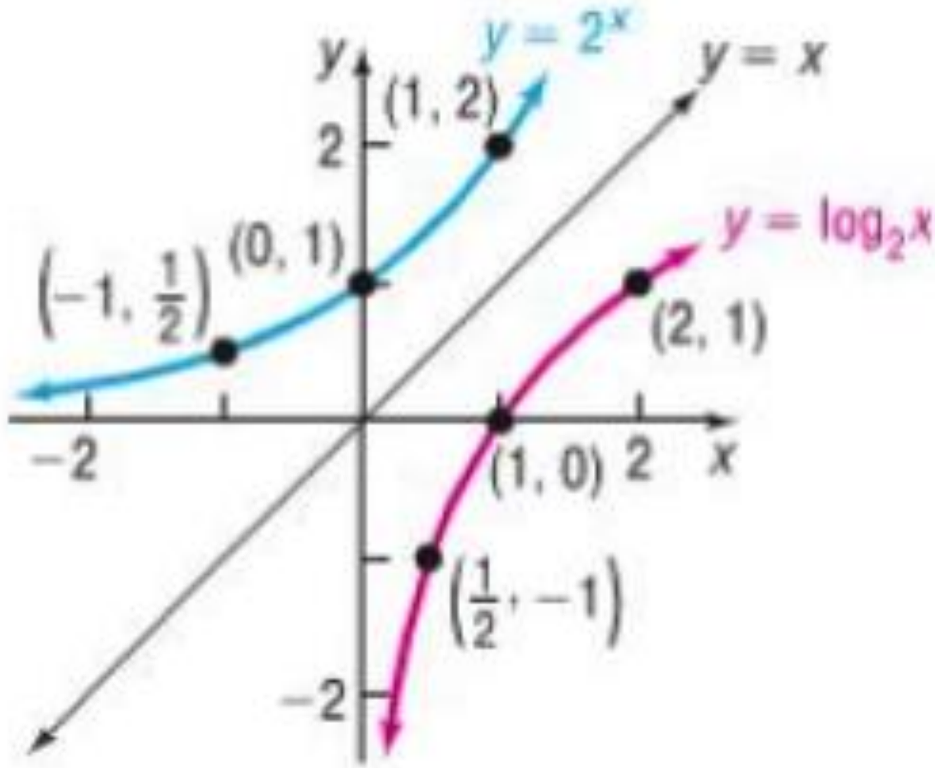


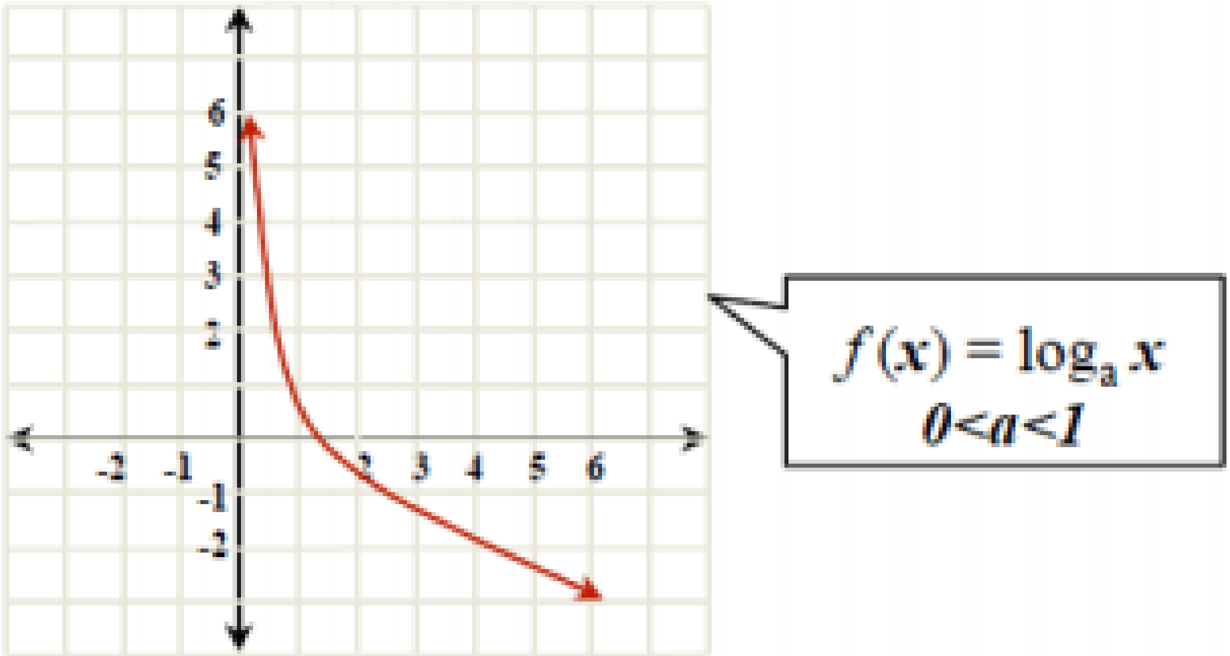
FIGURE 36

- $f(x) = \log_a x$, $a > 1$, is increasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph goes through the points $(a^{-1}, -1)$, $(1, 0)$, and $(a, 1)$.


y
x
3
−3
$y = a^x$
$y = x$
(1, a)
$y = \log_a x$
$\left(-1, \frac{1}{a}\right)$
(0, 1)
(a, 1)
(1, 0)
$\left(\frac{1}{a}, -1\right)$

(b) $a > 1$


y
x
2
−2
$y = 2^x$
$y = x$
(1, 2)
$y = \log_2 x$
$\left(-1, \frac{1}{2}\right)$
(0, 1)
(2, 1)
(1, 0)
$\left(\frac{1}{2}, -1\right)$



LOGARITHMIC FUNCTION $\quad f(x) = \log_a x, \quad 0 < a < 1$

Domain: $(0, \infty)$ $\qquad$ Range: $(-\infty, \infty)$

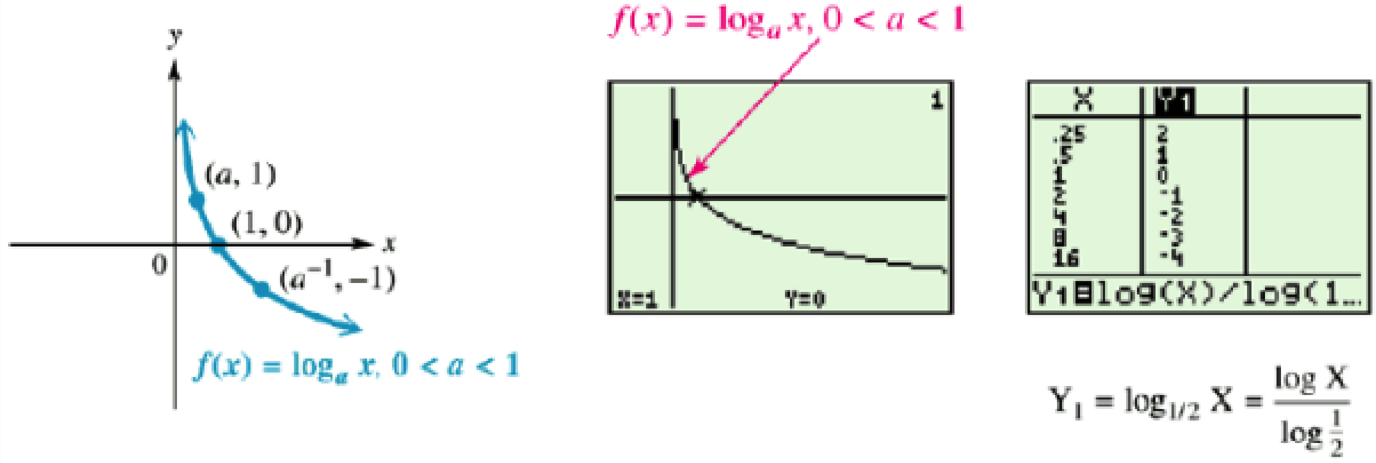
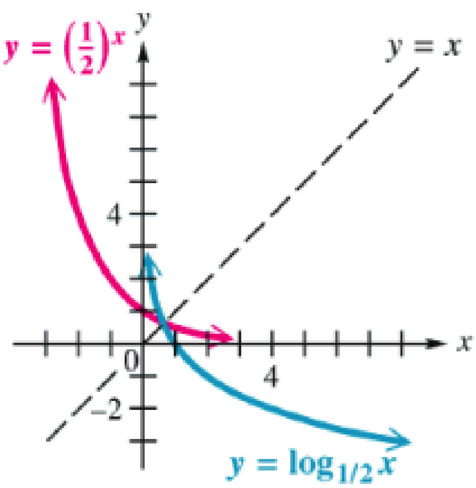


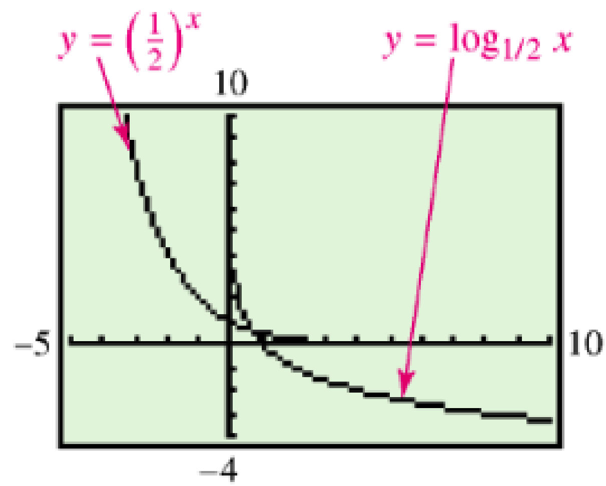
FIGURE 37

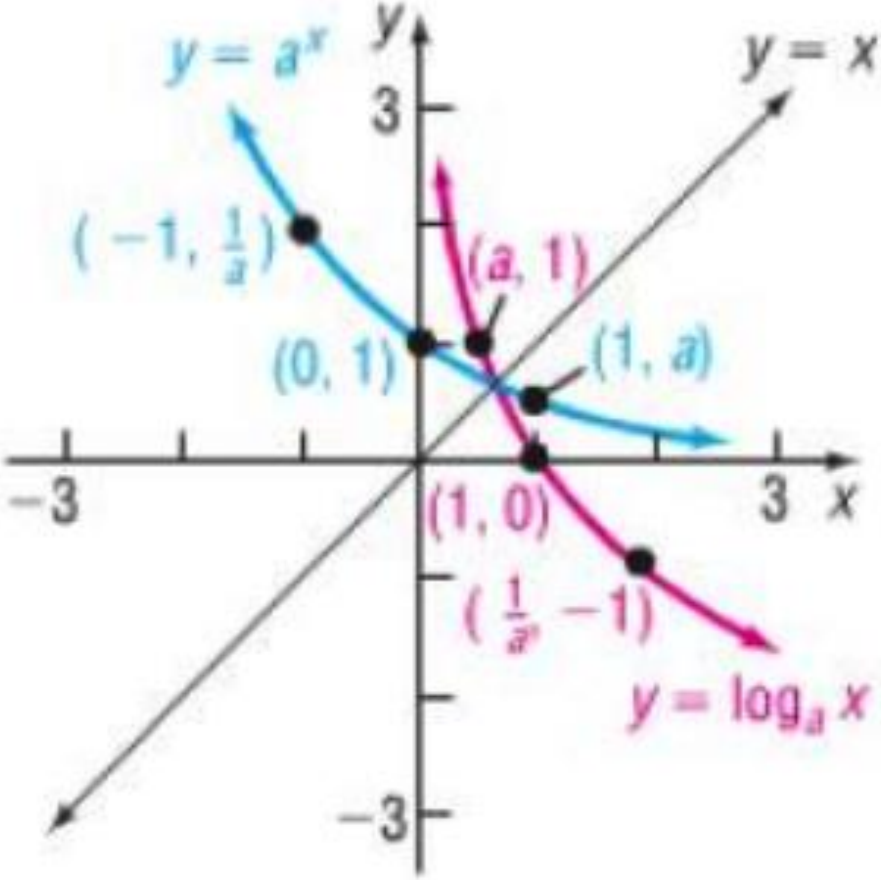
- $f(x) = \log_a x$, $0 < a < 1$, is decreasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph goes through the points $(a, 1)$, $(1, 0)$, and $(a^{-1}, -1)$.

Graphs Logs Func 0<a<1

- Below are typical shapes for such graphs where $0 < a < 1$

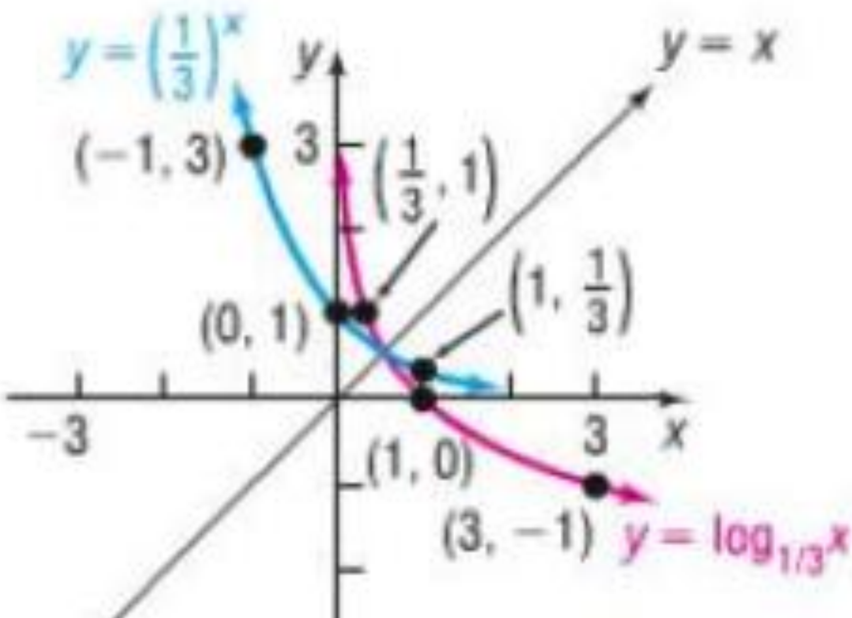






y = a^x
y
y = x
3
(−1, 1/a)
(a, 1)
(0, 1)
(1, a)
−3
3 x
(1, 0)
(1/a, −1)
y = log_a x
−3

(a) $0 < a < 1$

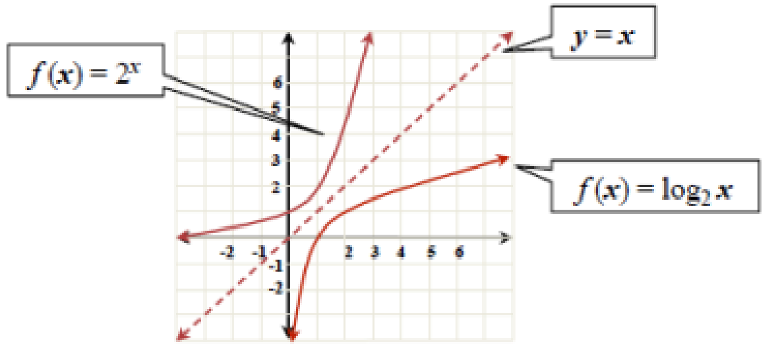


y = (1/3)^x
y
y = x
(−1, 3)
3
(1/3, 1)
(1, 1/3)
(0, 1)
−3
3
x
(1, 0)
(3, −1)
y = log_{1/3} x
−3

Graph $f(x) = 2^x$ and $g(x) = \log_2 x$ in the same rectangular coordinate system.

Solution

We now sketch the basic exponential graph. The graph of the inverse (logarithmic) can also be drawn by reflecting the graph of $f(x) = 2^x$ over the line y = x.



Graph $f(x) = 2^x$ and $g(x) = \log_2 x$ in the same rectangular coordinate system.

Solution We first set up a table of coordinates for $f(x) = 2^x$. Reversing these coordinates gives the coordinates for the inverse function, $g(x) = \log_2 x$.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	1/4	1/2	1	2	4	8

x	1/4	1/2	1	2	4	8
$g(x) = \log_2 x$	-2	-1	0	1	2	3

Reverse coordinates.

BASE e

Summary Logs Base "e" and ln

- $\log_e x$ means ln x
- These are called natural logarithms
- y = ln x is the inverse of $y = e^x$
- The domain of y = ln x is (0, ∞)
- The range is the interval (-∞, ∞)

Figure 33

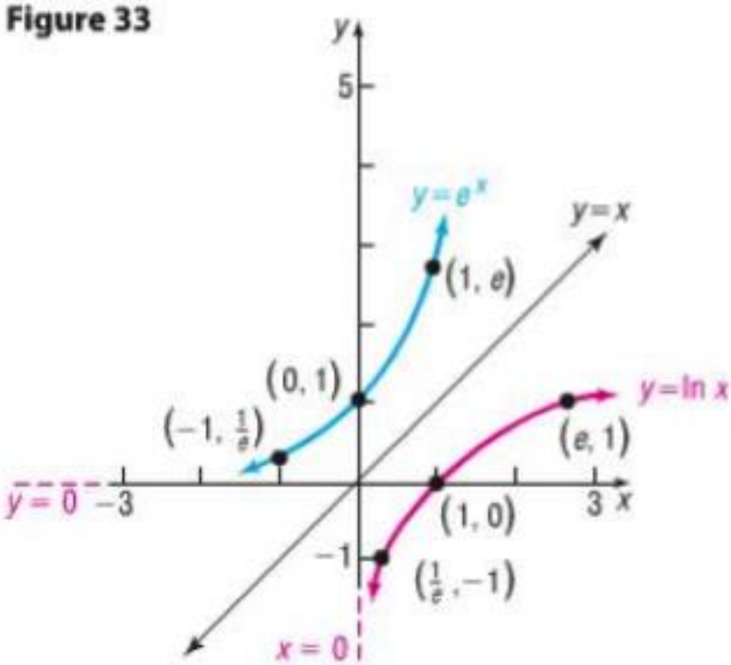


Table 7

x	ln x
$\frac{1}{2}$	−0.69
2	0.69
3	1.10

Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = -\ln(x - 2)$.

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.

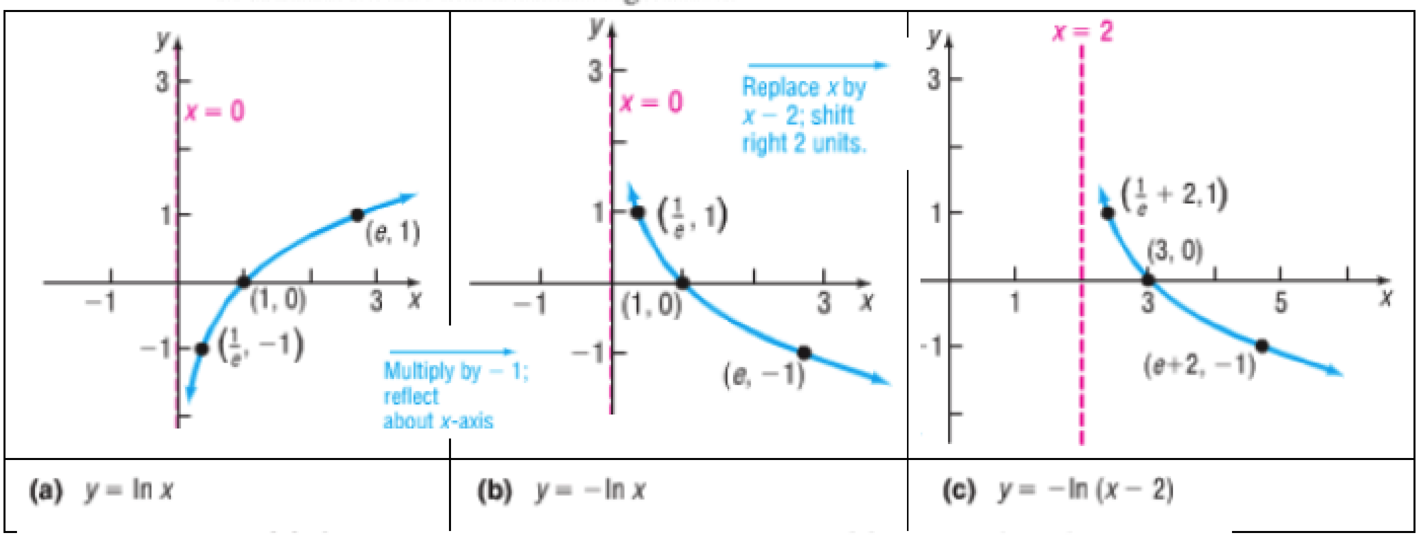
(d) Find f^{-1}, the inverse of f.

(e) Find the domain and the range of f^{-1}.

(f) Graph f^{-1}.

(a) The domain of f consists of all x for which $x - 2 > 0$ or, equivalently, $x > 2$. The domain of f is $\{x|x > 2\}$ or $(2, \infty)$ in interval notation.

(b) To obtain the graph of $y = -\ln(x - 2)$, we begin with the graph of $y = \ln x$ and use transformations. See Figure 34.



(c) The range of $f(x) = -\ln(x - 2)$ is the set of all real numbers. The vertical asymptote is $x = 2$. [Do you see why? The original asymptote ($x = 0$) is shifted to the right 2 units.]

(d) To find f^{-1}, begin with $y = -\ln(x - 2)$. The inverse function is defined (implicitly) by the equation

$$x = -\ln(y - 2)$$

Proceed to solve for y.

$-x = \ln(y - 2)$ Isolate the logarithm.

$e^{-x} = y - 2$ Change to an exponential statement.

$y = e^{-x} + 2$ Solve for y.

The inverse of f is $f^{-1}(x) = e^{-x} + 2$.

(e) The domain of f^{-1} equals the range of f, which is the set of all real numbers, from part (c). The range of f^{-1} is the domain of f, which is $(2, \infty)$ in interval notation.

(f) To graph f^{-1}, use the graph of f in Figure 34(c) and reflect it about the line $y = x$. See Figure 35. We could also graph $f^{-1}(x) = e^{-x} + 2$ using transformations.

including familiarity with the change of base formula to evaluate logarithms