LINES

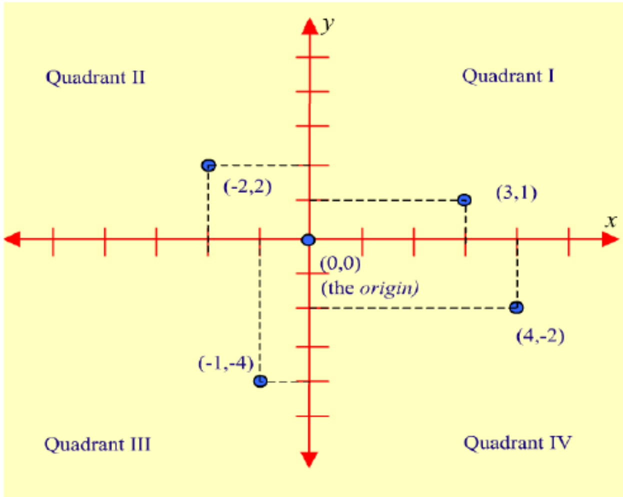
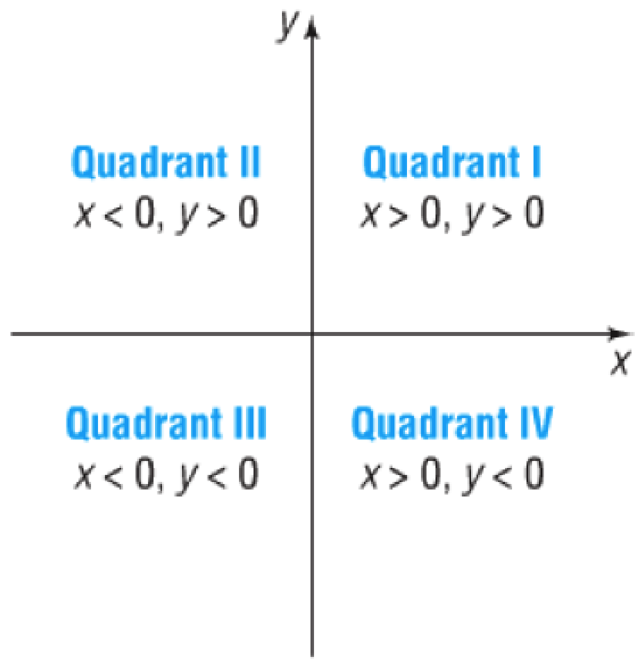


Figure 1: The Cartesian Plane $\mathbb{R}^2$



Distance Formula

Distance Formula

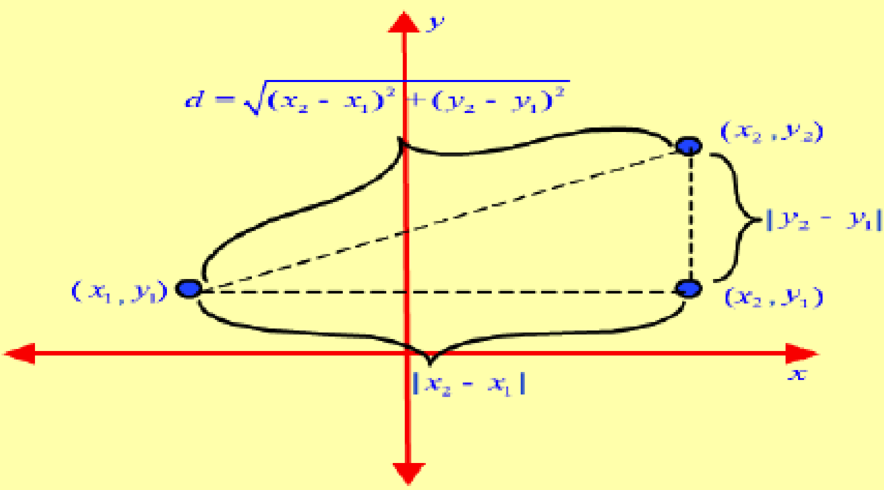
Letting (x_1, y_1) and (x_2, y_2) represent two points on the Cartesian plane, the distance between these two points may be found using the following formula, derived from the Pythagorean Theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let (x_1, y_1) and (x_2, y_2) be the coordinates of two arbitrary points in the plane. By drawing the dotted lines parallel to the coordinate axis as shown in Figure 2, we can form a right triangle. Note that we are able to determine the coordinates of the vertex at the right angle from the two vertices (x_1, y_1) and (x_2, y_2).



Find the distance d between the points $(-4, 5)$ and $(3, 2)$.

$$d = \sqrt{[3 - (-4)]^2 + (2 - 5)^2} = \sqrt{7^2 + (-3)^2}$$
$$= \sqrt{49 + 9} = \sqrt{58} \approx 7.62$$

Determine: **a.** the distance between (2, 0) and (–3, 4).

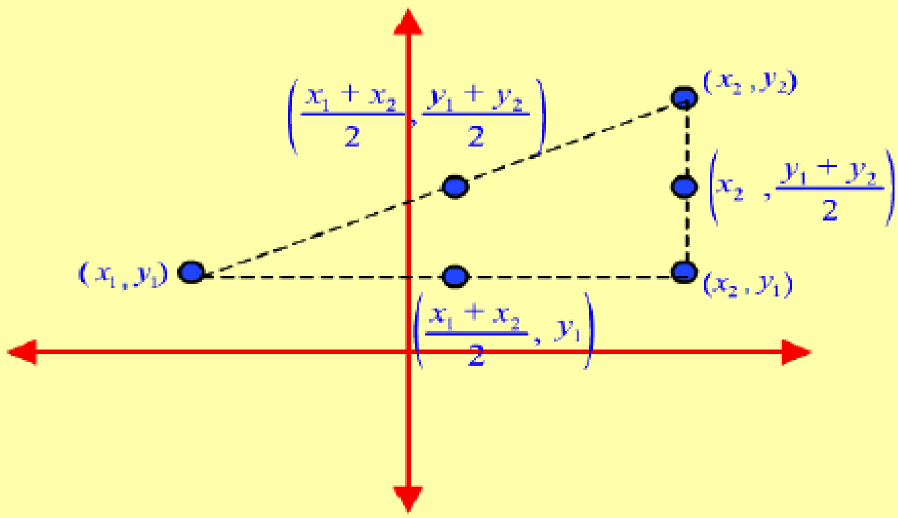
Midpoint Formula

Midpoint Formula

Letting (x_1, y_1) and (x_2, y_2) represent two points on the Cartesian plane, the midpoint between these two points may be found using the following formula, which finds the average of the two x-values and the average of the two y-values:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

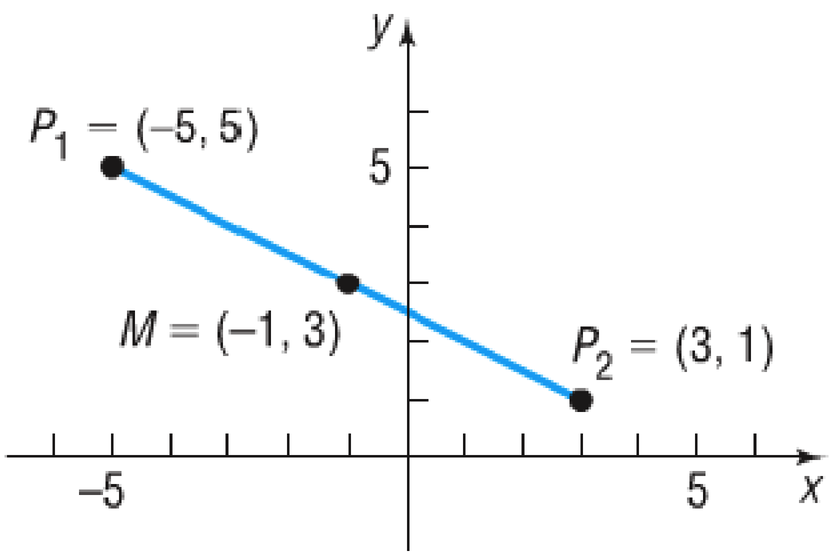
Consider the points as plotted in Figure 3. The x-coordinate of the midpoint should be the average of the two x-coordinates of the given points, and similarly for the y-coordinate.



Exercises

Find the midpoint of the line segment from $P_1 = (-5, 5)$ to $P_2 = (3, 1)$. Plot the points P_1 and P_2 and their midpoint.

$$x = \frac{x_1 + x_2}{2} = \frac{-5 + 3}{2} = -1 \quad \text{and} \quad y = \frac{y_1 + y_2}{2} = \frac{5 + 1}{2} = 3$$



Determine

b. the midpoint of the line segment joining (0, 2) and (–3, 4).

The Slope of a Line

Let L stand for a given line in the Cartesian plane, and let (x_1, y_1) and (x_2, y_2) be the coordinates of two distinct points on L. The **slope** of the line L is the ratio $\dfrac{y_2 - y_1}{x_2 - x_1}$, which can be described in words as "change in y over change in x" or "rise over run."

The phrase "rise over run" is motivated by the diagram in Figure 1.

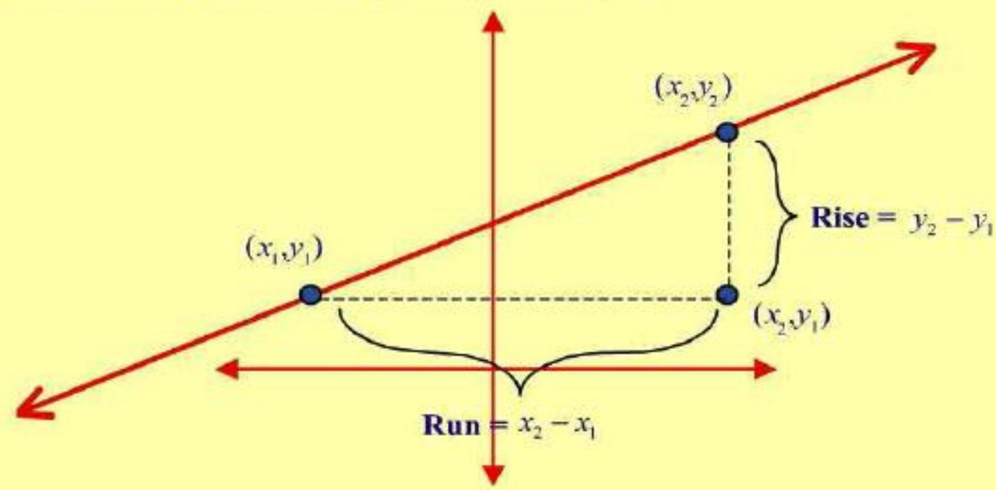
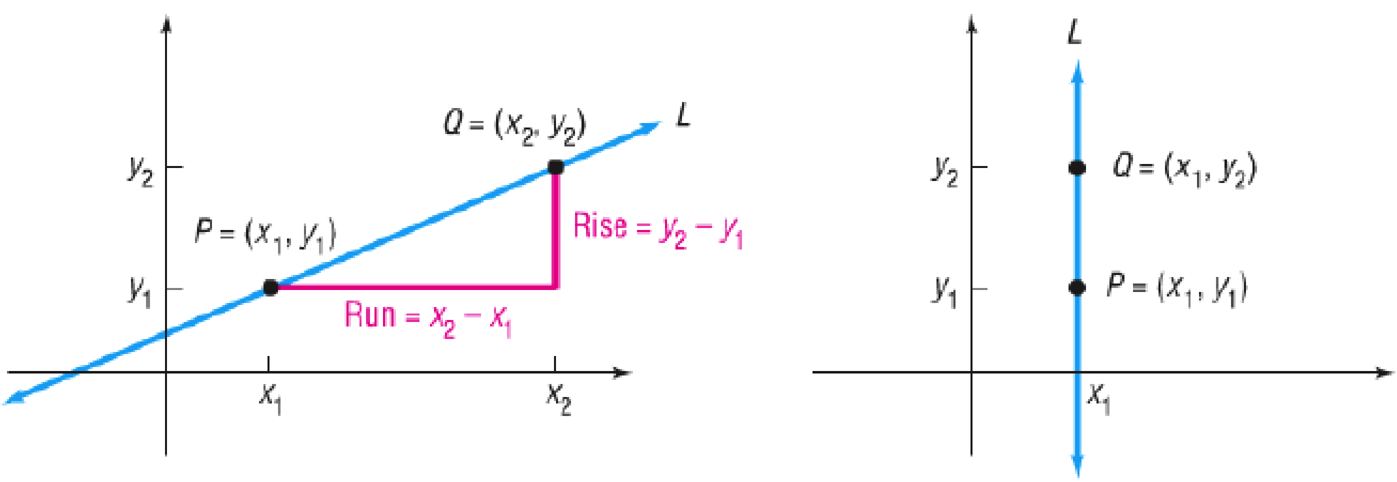


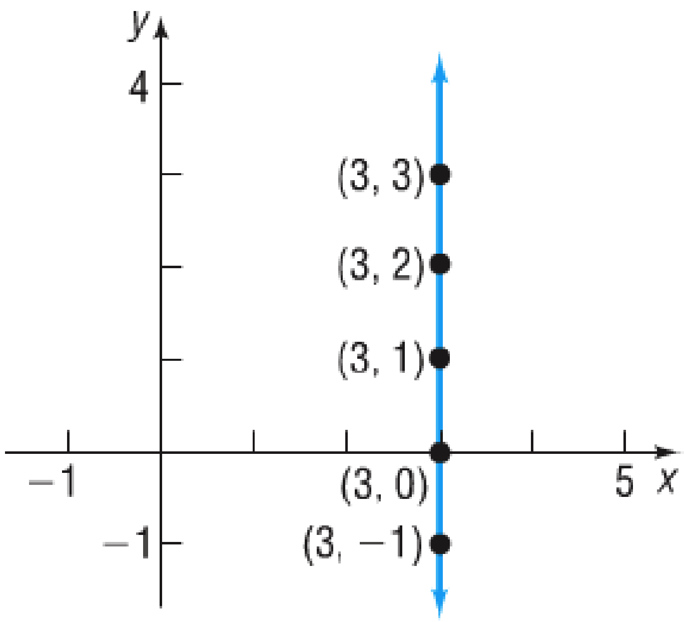
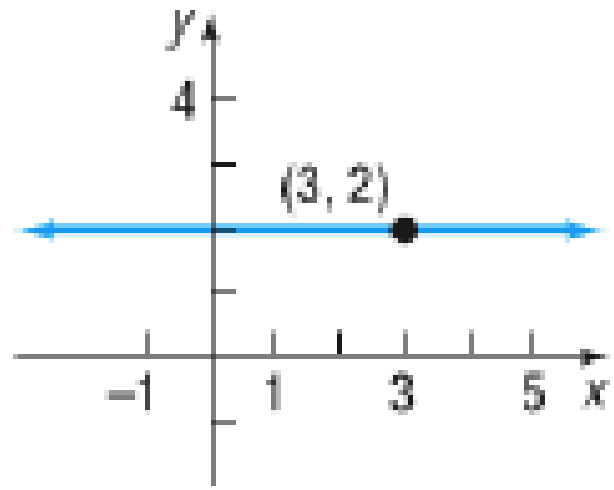
Figure 1: Rise and Run Between Two Points

As drawn in Figure 1, the ratio $\dfrac{y_2 - y_1}{x_2 - x_1}$ is positive, and we say that the line has positive slope. If the rise and run have opposite sign, the slope of the line would be negative and the line under consideration would be falling from the upper left to the lower right.



(a) Slope of L is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

(b) Slope is undefined; L is vertical

<table>
<tr><td>Graph the equation x = 3</td><td>Y = 2</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>What is the slope of this line?

What would be the perpendicular line to the point (3,2)?</td><td>What is the slope of this line?

What would be the perpendicular line to the point (3,2)?</td></tr>
</table>

Point Slope form of the line

Point-Slope Form of a Line

Given an ordered pair (x_1, y_1) and a real number m, an equation for the line passing through the point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$. Note that m, x_1, and y_1 are all constants, and that x and y are variables. Note also that since the line, by definition, has slope m, vertical lines cannot be described in this form.

Find an equation of the horizontal line containing the point $(3, 2)$.

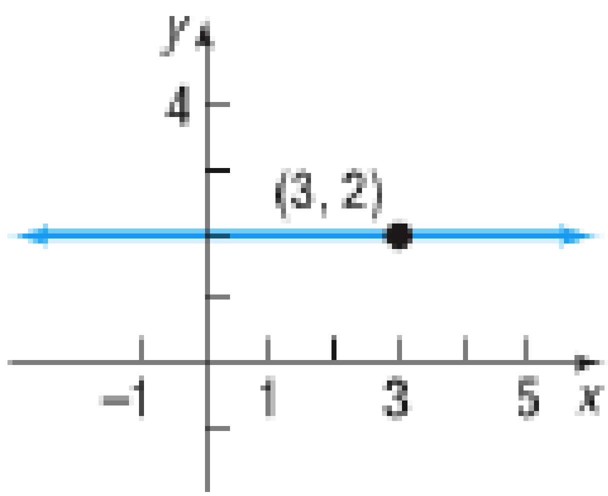
Because all the y-values are equal on a horizontal line, the slope of a horizontal line is 0. To get an equation, we use the point–slope form with $m = 0$, $x_1 = 3$, and $y_1 = 2$.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 0 \cdot (x - 3) \qquad m = 0,\ x_1 = 3, \text{ and } y_1 = 2$$

$$y - 2 = 0$$

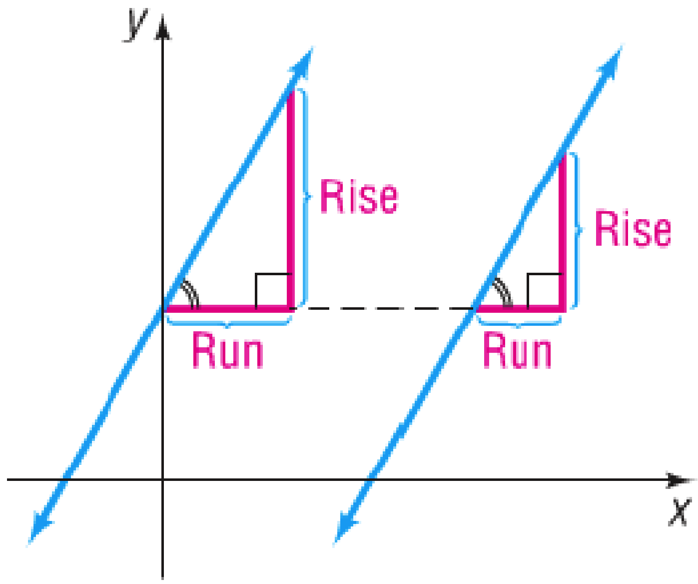
$$y = 2$$



Parallel Lines

THEOREM **Criterion for Parallel Lines**

Two nonvertical lines are parallel if and only if their slopes are equal and they have different y-intercepts.



EXAMPLE 10 **Showing That Two Lines Are Parallel**

Show that the lines given by the following equations are parallel:

$$L_1:\quad 2x + 3y = 6, \qquad L_2:\quad 4x + 6y = 0$$

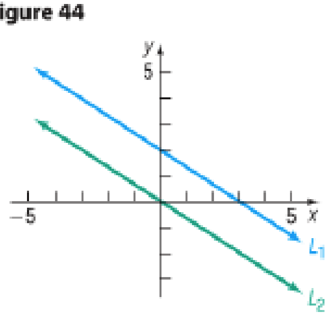
Solution To determine whether these lines have equal slopes and different y-intercepts, write each equation in slope–intercept form:

$$L_1:\quad 2x + 3y = 6 \qquad\qquad L_2:\quad 4x + 6y = 0$$
$$3y = -2x + 6 \qquad\qquad 6y = -4x$$
$$y = -\frac{2}{3}x + 2 \qquad\qquad y = -\frac{2}{3}x$$

$$\text{Slope} = -\frac{2}{3};\ y\text{-intercept} = 2 \qquad \text{Slope} = -\frac{2}{3};\ y\text{-intercept} = 0$$

Because these lines have the same slope, $-\frac{2}{3}$, but different y-intercepts, the lines are parallel. See Figure 44.

Figure 44



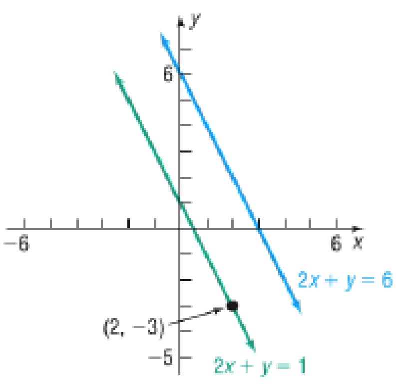
EXAMPLE 11 Finding a Line That Is Parallel to a Given Line

Find an equation for the line that contains the point $(2, -3)$ and is parallel to the line $2x + y = 6$.

Solution Since the two lines are to be parallel, the slope of the line that we seek equals the slope of the line $2x + y = 6$. Begin by writing the equation of the line $2x + y = 6$ in slope–intercept form.

$$\begin{aligned} 2x + y &= 6 \\ y &= -2x + 6 \end{aligned}$$

Figure 45



The slope is -2. Since the line that we seek also has slope -2 and contains the point $(2, -3)$, use the point–slope form to obtain its equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point–slope form} \\ y - (-3) &= -2(x - 2) && m = -2, x_1 = 2, y_1 = -3 \\ y + 3 &= -2x + 4 && \text{Simplify.} \\ y &= -2x + 1 && \text{Slope–intercept form} \\ 2x + y &= 1 && \text{General form} \end{aligned}$$

This line is parallel to the line $2x + y = 6$ and contains the point $(2, -3)$. See Figure 45.

Now Work PROBLEM 59

Perpendicular Lines

Slope of Perpendicular Lines

If m_1 and m_2 represent the slopes of two perpendicular lines, neither of which is vertical,

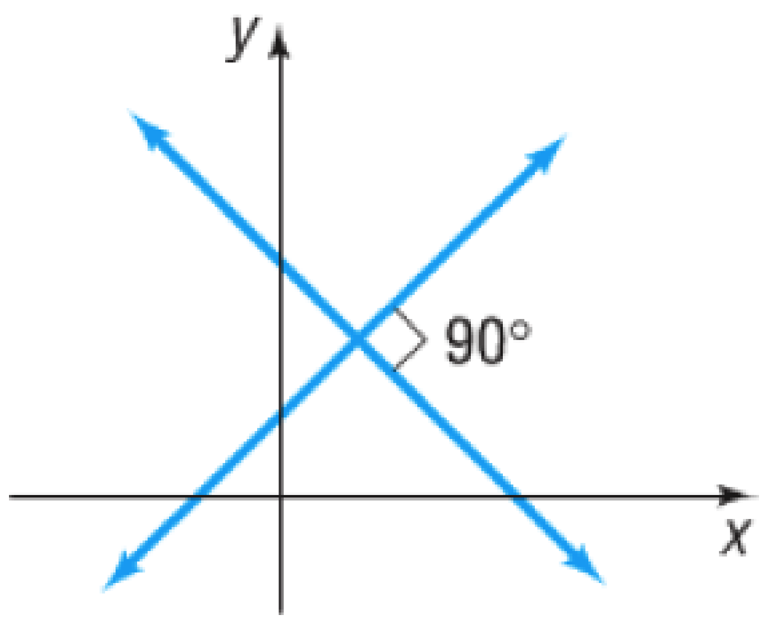
$$m_1 = \frac{-1}{m_2} \quad \text{and} \quad m_2 = \frac{-1}{m_1}$$

if one of the two perpendicular lines is vertical, the other is horizontal, and the slopes are, respectively, undefined and zero.

THEOREM

Criterion for Perpendicular Lines

Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.



Please note how perpendicular lines form 90 degree angles at their intersection.

Finding the Equation of a Line Perpendicular to a Given Line

Find an equation of the line that contains the point $(1, -2)$ and is perpendicular to the line $x + 3y = 6$. Graph the two lines.

First write the equation of the given line in slope–intercept form to find its slope.

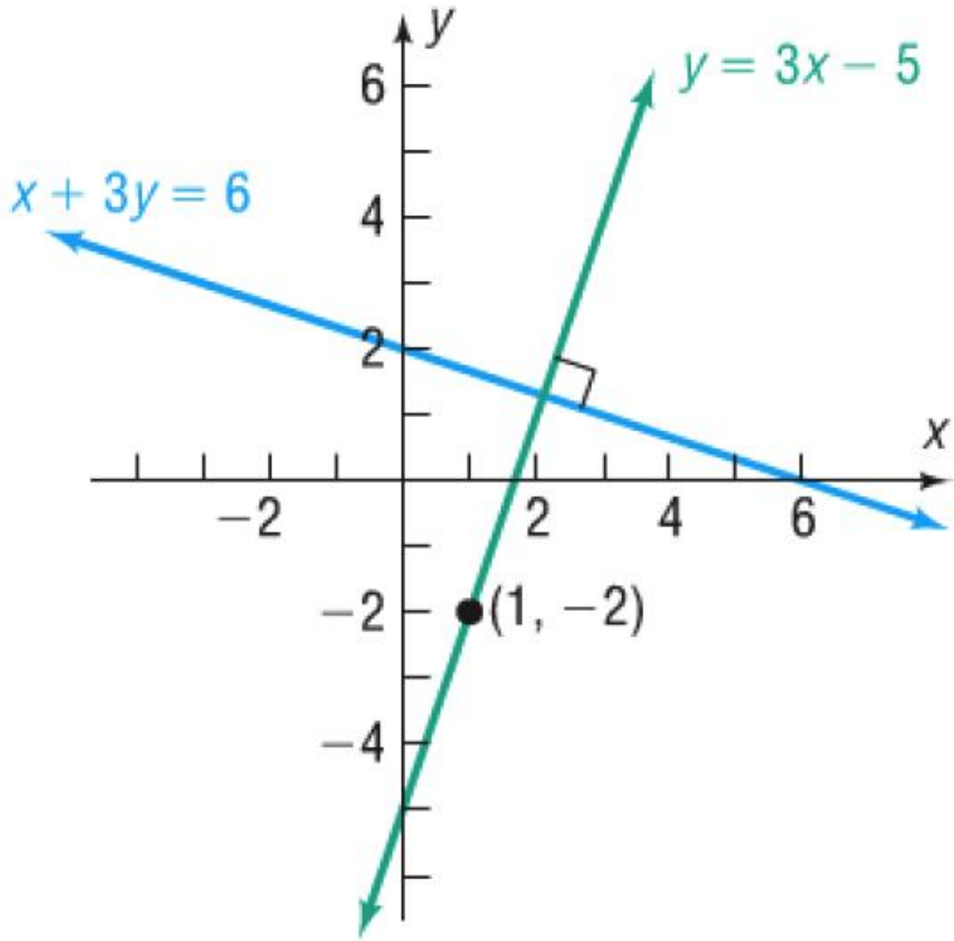
$$\begin{aligned} x + 3y &= 6 \\ 3y &= -x + 6 && \text{Proceed to solve for } y. \\ y &= -\frac{1}{3}x + 2 && \text{Place in the form } y = mx + b. \end{aligned}$$

The given line has slope $-\frac{1}{3}$. Any line perpendicular to this line will have slope 3. Because we require the point $(1, -2)$ to be on this line with slope 3, use the point–slope form of the equation of a line.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point–slope form} \\ y - (-2) &= 3(x - 1) && m = 3, x_1 = 1, y_1 = -2 \end{aligned}$$

To obtain other forms of the equation, proceed as follows:

$$\begin{aligned} y + 2 &= 3x - 3 && \text{Simplify.} \\ y &= 3x - 5 && \text{Slope–intercept form} \\ 3x - y &= 5 && \text{General form} \end{aligned}$$



y
x
$y = 3x - 5$
$x + 3y = 6$
6
4
2
−2
−4
−2
2
4
6
(1, −2)

Exercises

Consider the following equation. $3y - 24 = 4x$

(a) Determine the x- and y- intercepts of the given equation. If one of the intercepts does not exist, state "absent" for that intercept.

(b) Graph the given equation by plotting the x and y intercepts. If an intercept does not exist, use another point to plot the graph.

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For the points $A = (-6, -1)$ and $B = (2, -10)$.

(a) Find the distance between A and B.

(b) Find the coordinates of the midpoint.

(c) Determine the slope of the line that passes through A and B. Please enter your answer in simplest form. If the slope is undefined state "Undefined".

//

Find the slope of the line determined by the following equations. Please enter your answer in simplest form. If the slope is undefined state "Undefined".

(a) $5y + 3x = 7$

(b) $4y = 8$

(c) $3x - 1 = 0$

(d) $y = 4x - 1$

//

Consider the following equation. $x + 4y = 5$

(a) Rewrite the equation in slope-intercept form.

(b) Given $x = -7$, find the value for y and graph.

(c) Given $x = -3$, find the value for y and use the points to complete the graph of the line.

//

Write the slope-intercept form of the equation for the line that passes through the points $(-6, 3)$ and $(1, 4)$.

//

Consider the following equation of a line. Reduce all fractions to lowest terms. $8x + 4y = 15$

(a) Rewrite this equation in slope-intercept form.

(b) Find the equation, in slope-intercept form, for the line which is parallel to this line and passes through the point $(-7, 8)$.

//

Complete the sentences below:

The line $y = 5x + 2$ and $y = ax - 1$ are **perpendicular** if $a =$ ________________.

The line $y = 3x - 1$ and $y = ax$ are **parallel** if $a =$ ________________.

//

The slope of a **vertical** line is ____________ ; the slope of a **horizontal** line is ____________.

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