Exponential Functions, Graphs & Applications

Section 5.1, 5.2

Math 120

Definition of the Exponential Function

The exponential function f with base b is defined by

$$f(\mathrm{x}) = b^x \quad \text{or} \quad \boldsymbol{y} = b^x$$

Where b is a positive constant other than and $\boldsymbol{x}$ is any real number.

Here are some examples of exponential functions.

$f(\boldsymbol{x}) = 2^x$ — Base is 2.

$g(\boldsymbol{x}) = 10^x$ — Base is 10.

$h(\boldsymbol{x}) = 3^{x+1}$ — Base is 3.

The value of $f(x) = 3^x$ when $x = 2$ is

$$f(2) = 3^2 = 9$$

The value of $f(x) = 3^x$ when $x = -2$ is

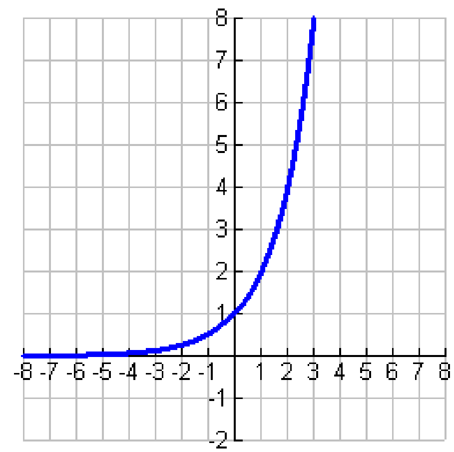
$$f(-2) = 3^{-2} = \frac{1}{9}$$

The value of $g(x) = 0.5^x$ when $x = 4$ is

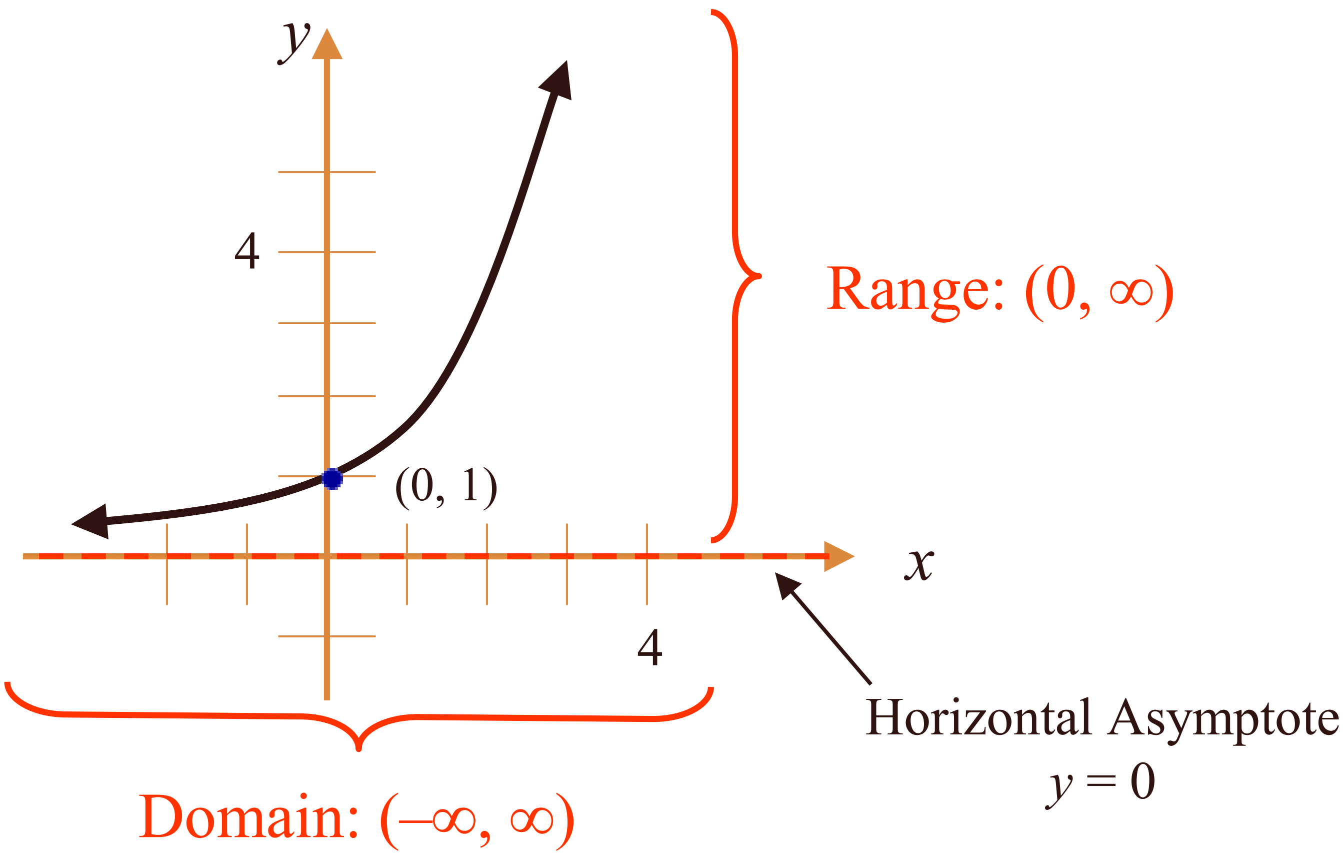
$$g(4) = 0.5^4 = 0.0625$$

Definition of Exponential Functions

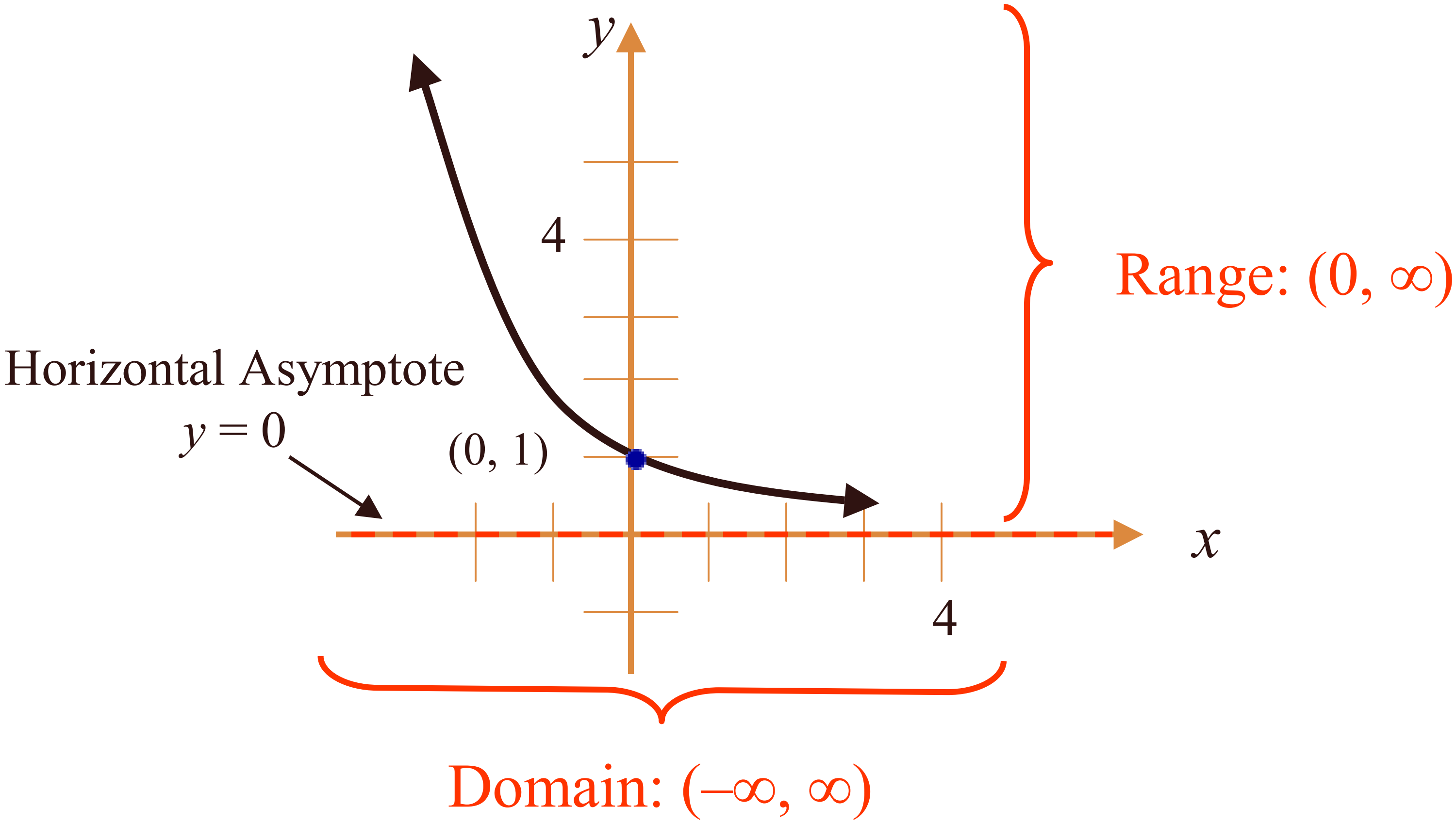
- The exponential function *f* with a base *b* is defined by $f(x) = b^x$ where b is a positive constant other than 1 ($b > 0$, and $b \neq 1$) and x is any real number.
- So, $f(x) = 2^x$, looks like:



The graph of $f(x) = a^x$, $a > 1$

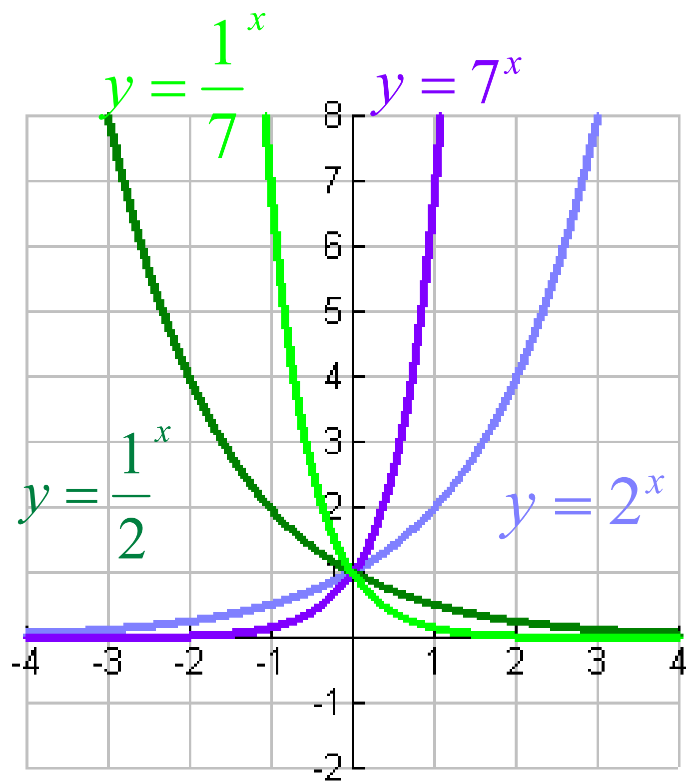


The graph of $f(x) = a^x$, $0 < a < 1$



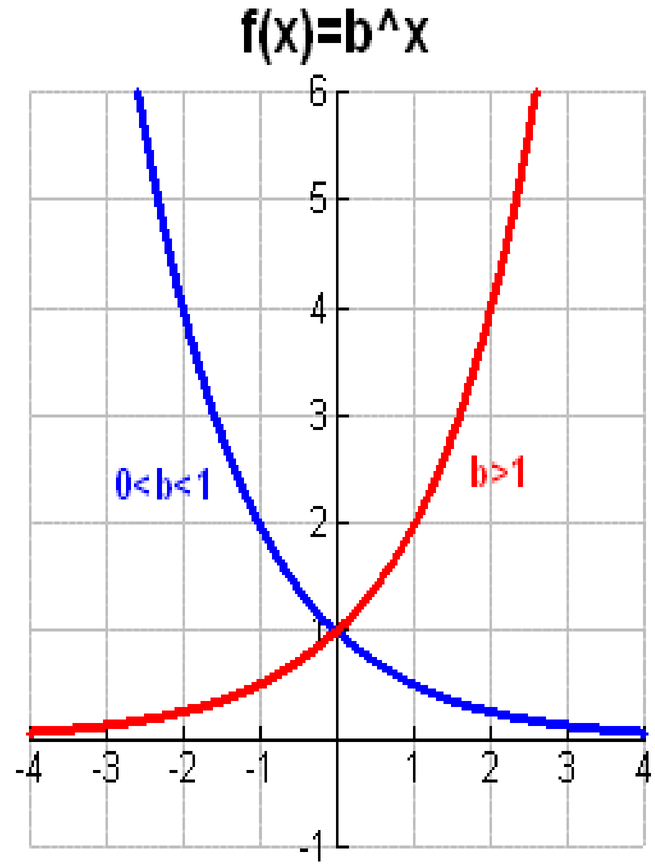
Graphing Exponential Functions

- Four exponential functions have been graphed. Compare the graphs of functions where b > 1 to those where b < 1



Graphing Exponential Functions

- So, when b > 1, f(x) has a graph that goes up to the right and is an increasing function.
- When 0 < b < 1, f(x) has a graph that goes down to the right and is a decreasing function.



Characteristics

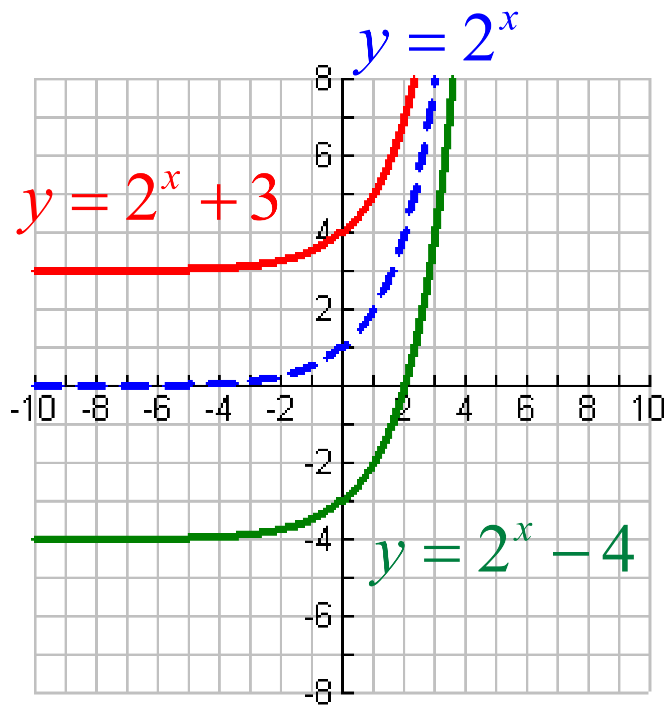
- The domain of $f(x) = b^x$ consists of all real numbers $(-\infty, \infty)$. The range of $f(x) = b^x$ consists of all positive real numbers $(0, \infty)$.
- The graphs of all exponential functions pass through the point (0,1). This is because $f(o) = b^0 = 1$ $(b \neq o)$.
- The graph of $f(x) = b^x$ approaches but does not cross the x-axis. The x-axis is a horizontal asymptote.
- $f(x) = b^x$ is one-to-one and has an inverse that is a function.

Transformations Defined

Transformation	Equation	Description
Horizontal translation	$g(x) = b^{x+c}$	• Shifts the graph of $f(x) = b^x$ to the left c units if $c > 0$. • Shifts the graph of $f(x) = b^x$ to the right c units if $c < 0$.
Vertical stretching or shrinking	$g(x) = c\,b^x$	Multiplying y-coordintates of $f(x) = b^x$ by c, • Stretches the graph of $f(x) = b^x$ if $c > 1$. • Shrinks the graph of $f(x) = b^x$ if $0 < c < 1$.
Reflecting	$g(x) = -b^x$ $g(x) = b^{-x}$	• Reflects the graph of $f(x) = b^x$ about the x-axis. • Reflects the graph of $f(x) = b^x$ about the y-axis.
Vertical translation	$g(x) = -b^x + c$	• Shifts the graph of $f(x) = b^x$ upward c units if $c > 0$. • Shifts the graph of $f(x) = b^x$ downward c units if $c < 0$.

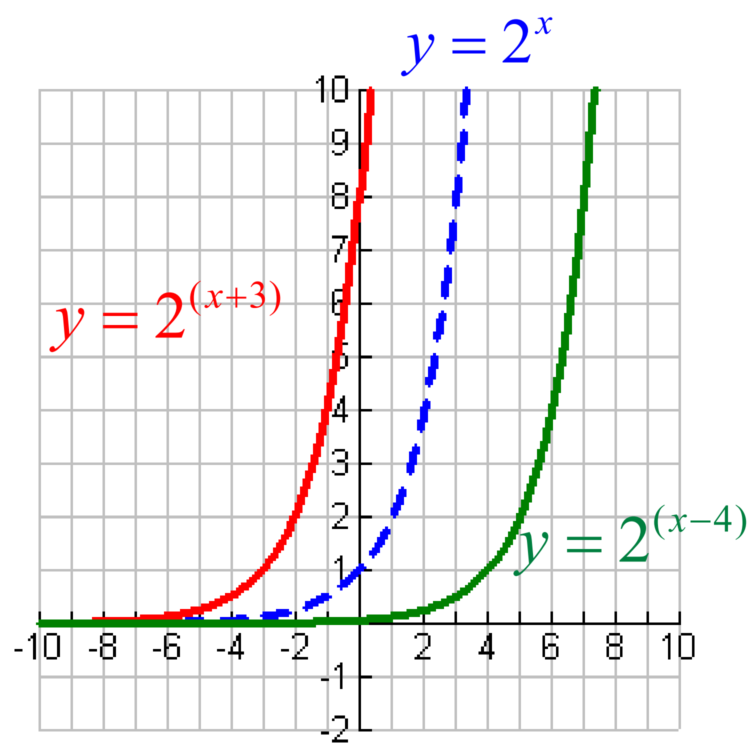
Transformations

- **Vertical translation**
  $f(x) = b^x + c$
- Shifts the graph up if $c > 0$
- Shifts the graph down if $c < 0$



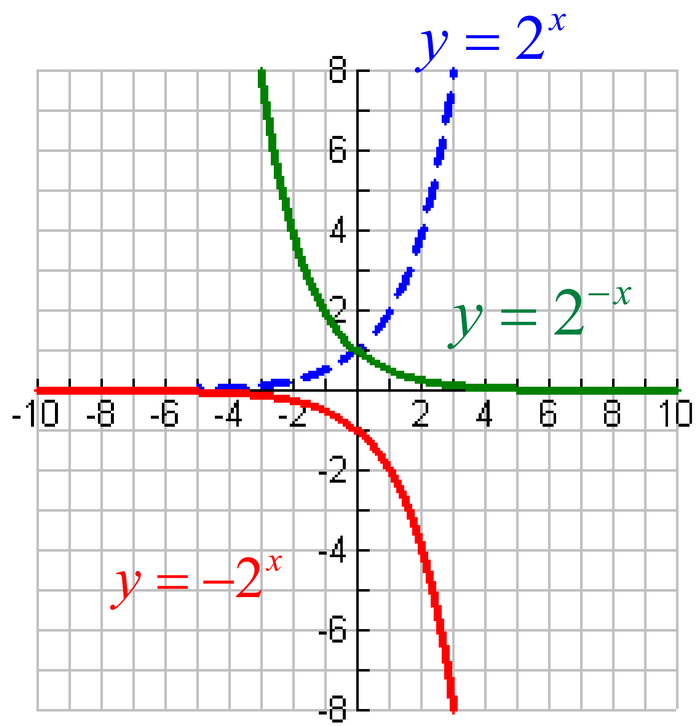
Transformations

- **Horizontal translation: $g(x)=b^{x+c}$**
- Shifts the graph to the left if $c > 0$
- Shifts the graph to the right if $c < 0$



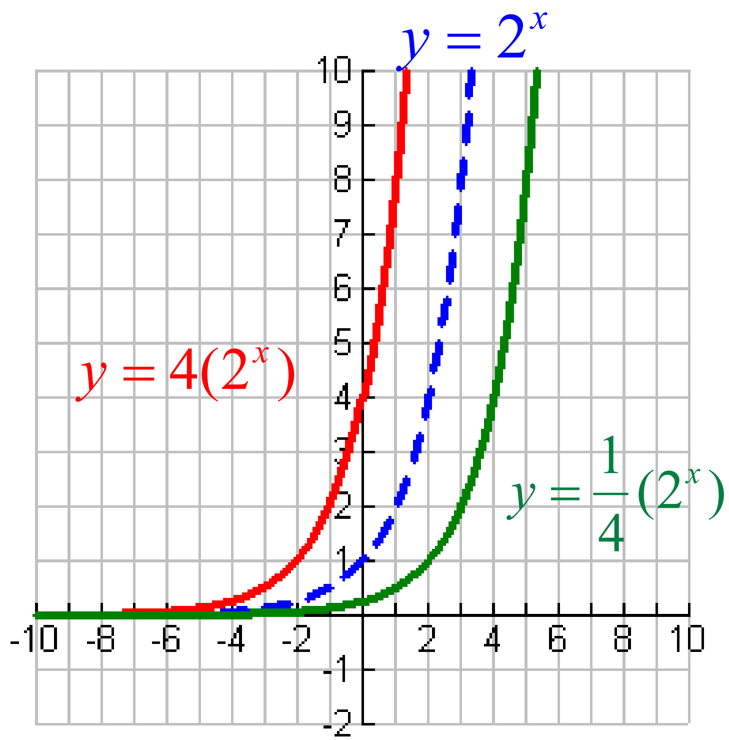
Transformations

- **Reflecting**
- $g(x) = -b^x$ reflects the graph about the x-axis.
- $g(x) = b^{-x}$ reflects the graph about the y-axis.



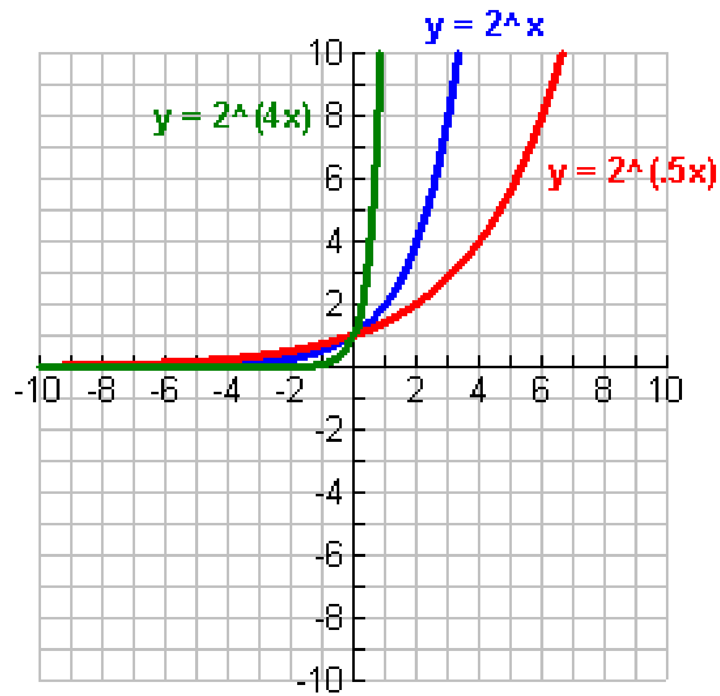
Transformations

- **Vertical stretching or shrinking, $f(x)=cb^x$:**
- Stretches the graph if c > 1
- Shrinks the graph if 0 < c < 1



Transformations

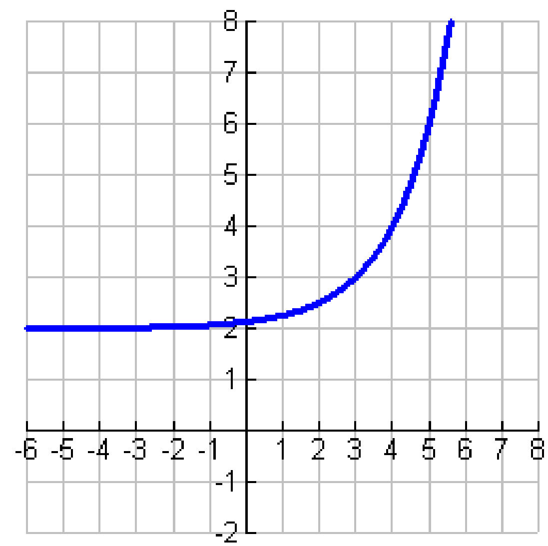
- **Horizontal stretching or shrinking**, **$f(x)=b^{cx}$:**
- Shinks the graph if $c > 1$
- Stretches the graph if $0 < c < 1$



You Do

- Graph the function $f(x) = 2^{(x-3)} + 2$
- Where is the horizontal asymptote?

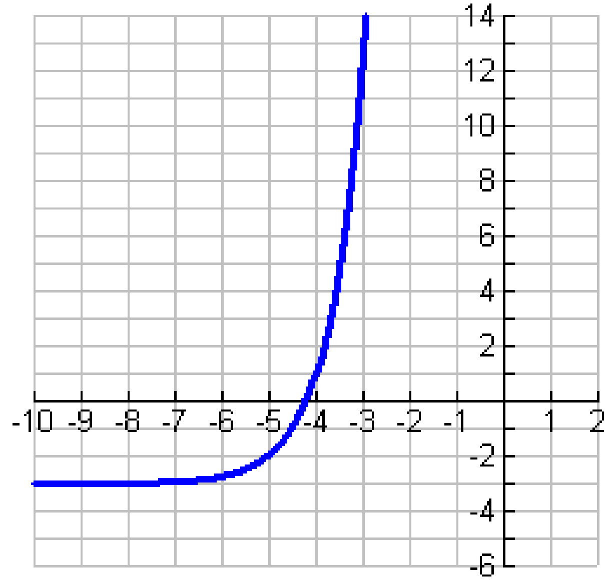
$y = 2$



You Do, Part Deux

- Graph the function $f(x) = 4^{(x+5)} - 3$
- Where is the horizontal asymptote?

y = - 3



The Number e

- The number e is known as Euler’s number. Leonard Euler (1700’s) discovered it’s importance.
- The number *e* has physical meaning. It occurs naturally in any situation where a quantity increases at a rate proportional to its value, such as a bank account producing interest, or a population increasing as its members reproduce.

The Number e - Definition

- An irrational number, symbolized by the letter e, appears as the base in many applied exponential functions. It models a variety of situations in which a quantity grows or decays continuously: money, drugs in the body, probabilities, population studies, atmospheric pressure, optics, and even spreading rumors!

- The number e is defined as the value that $\left(1+\frac{1}{n}\right)^n$ approaches as n gets larger and larger.

The Number e - Definition

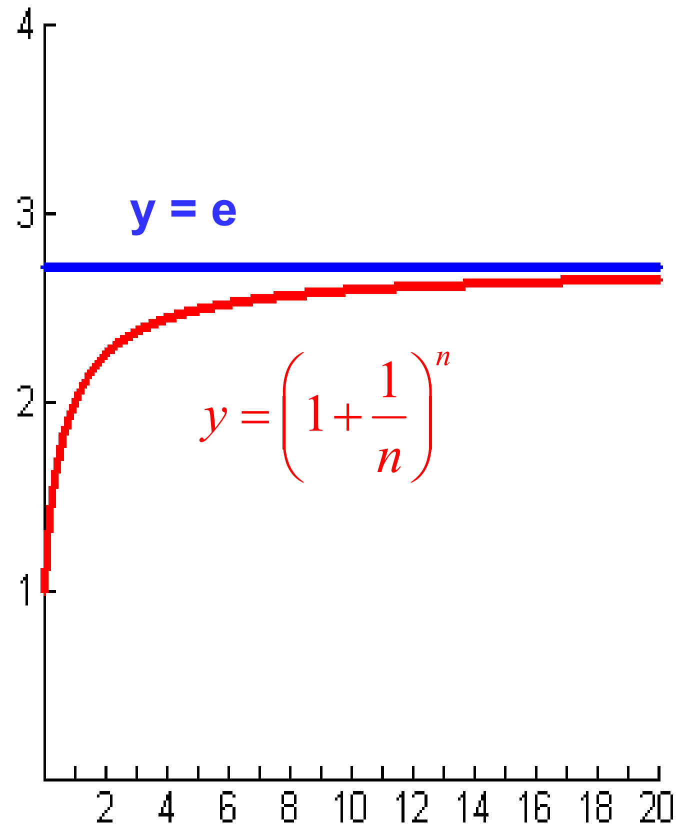
The table shows the values of $A_0\left(1+\frac{1}{n}\right)^n$ as n gets increasingly large.

As $n \to \infty$, the approximate value of e (to 9 decimal places) is ≈ 2.718281827

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829
1000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
1,000,000,000	2.718281827
$As\ n \to \infty,\ \left(1+\frac{1}{n}\right)^n \to e$	

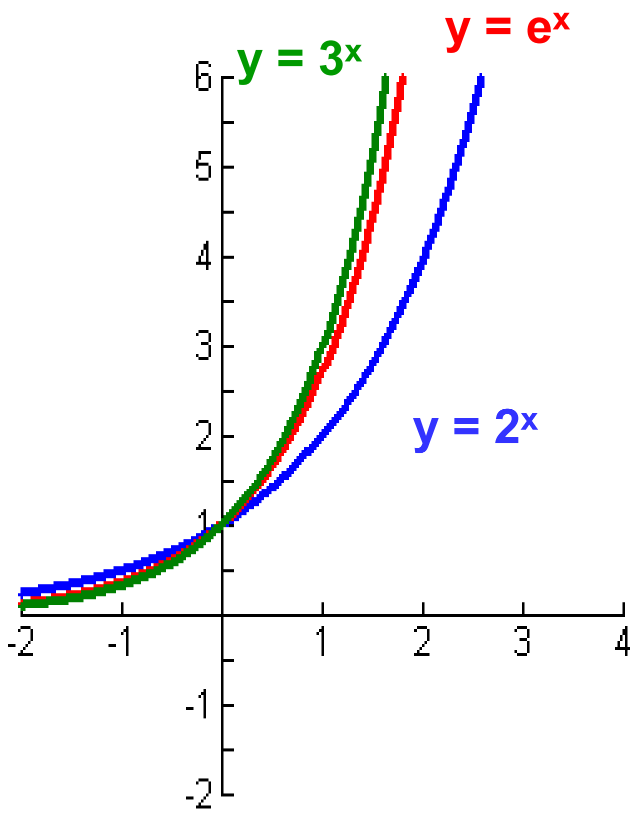
The Number e - Definition

- For our purposes, we will use $e \approx 2.718$.
- e is 2nd function on the division key on your calculator.



The Number e - Definition

- Since $2 < e < 3$, the graph of $y = e^x$ is between the graphs of $y = 2^x$ and $y = 3^x$



- e^x is the 2nd function on the ln key on your calculator

Natural Base

- The irrational number e, is called the natural base.
- The function $f(x) = e^x$ is called the natural exponential function.

Compound Interest

- The formula for compound interest:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where n is the number of times per year interest is being compounded and r is the *annual* rate.

Compound Interest

- Consider an amount A_0 of money deposited in an account
  - Pays annual rate of interest *r* percent
  - Compounded *m* times per year
  - Stays in the account *n* years
- Then the resulting balance A_n

$$A_n = A_0 \left(1 + \frac{r}{m}\right)^{m \cdot n}$$

Compound Interest - Example

- Which plan yields the most interest?
  - Plan A: A $1.00 investment with a 7.5% annual rate compounded monthly for 4 years
  - Plan B: A $1.00 investment with a 7.2% annual rate compounded daily for 4 years
  - A: $$1\left(1+\frac{0.075}{12}\right)^{12(4)} \approx 1.3486$$

    $1.35

    
  - B: $$1\left(1+\frac{0.072}{365}\right)^{365(4)} \approx 1.3337$$

    $1.34

Interest Compounded Continuously

- If interest is compounded "all the time" (MUST use the word continuously), we use the formula

$$A(t) = Pe^{rt}$$

where P is the initial principle (initial amount)

$$A(t) = Pe^{rt}$$

- If you invest $1.00 at a 7% annual rate that is compounded continuously, how much will you have in 4 years?

$$1 * e^{(.07)(4)} \approx 1.3231$$

- You will have a whopping $1.32 in 4 years!

You Do

- You decide to invest $8000 for 6 years and have a choice between 2 accounts. The first pays 7% per year, compounded monthly. The second pays 6.85% per year, compounded continuously. Which is the better investment?

You Do Answer

- 1st Plan:

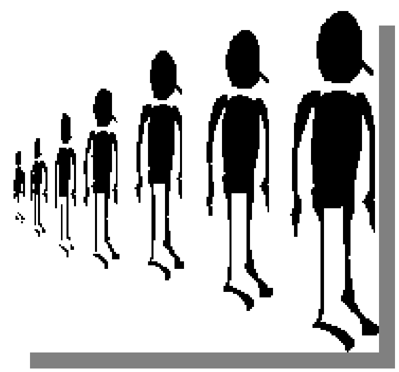
$$A(6) = 8000\left(1 + \frac{0.07}{12}\right)^{12(6)} \approx \$12{,}160.84$$

- 2nd Plan:

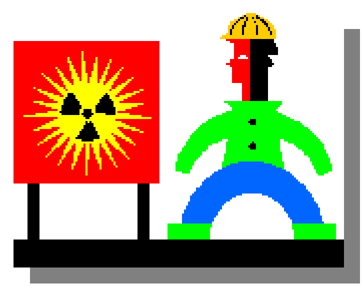
$$P(6) = 8000e^{0.0685(6)} \approx \$12{,}066.60$$

Exponential Modeling

- Population growth often modeled by exponential function



- Half life of radioactive materials modeled by exponential function



Growth Factor

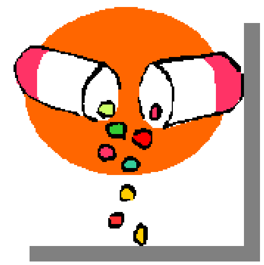
- Recall formula

  new balance = old balance + 0.05 * old balance
- Another way of writing the formula

  new balance = 1.05 * old balance
- Why equivalent?
- Growth factor: 1 + interest rate as a fraction

Decreasing Exponentials

- Consider a medication
  - Patient takes 100 mg
  - Once it is taken, body filters medication out over period of time
  - Suppose it removes 15% of what is present in the blood stream every hour



At end of hour	Amount remaining
1	100 – 0.15 * 100 = 85
2	85 – 0.15 * 85 = 72.25
3	
4	
5	

Fill in the rest of the table

What is the growth factor?

Decreasing Exponentials

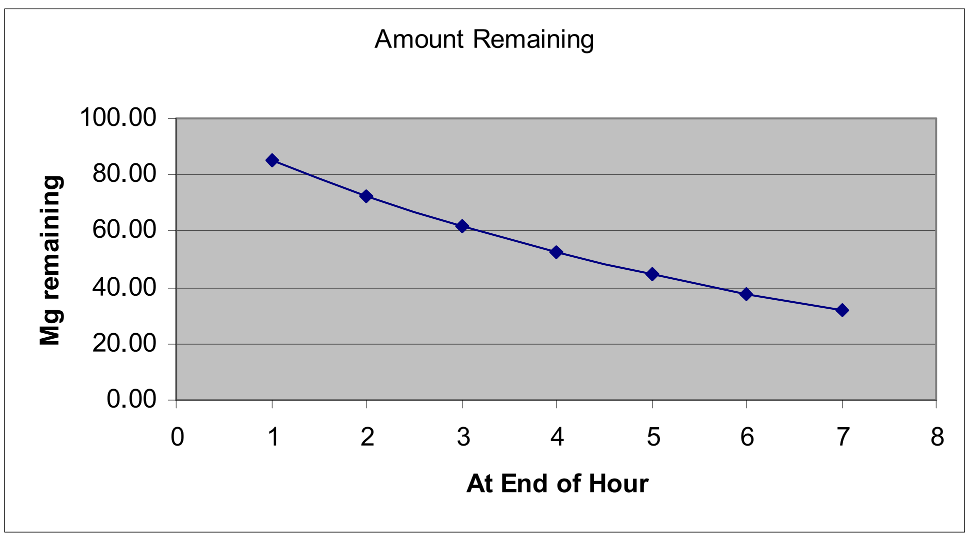
- Completed chart

Growth Factor = 0.85

Note: when growth factor < 1, exponential is a decreasing function

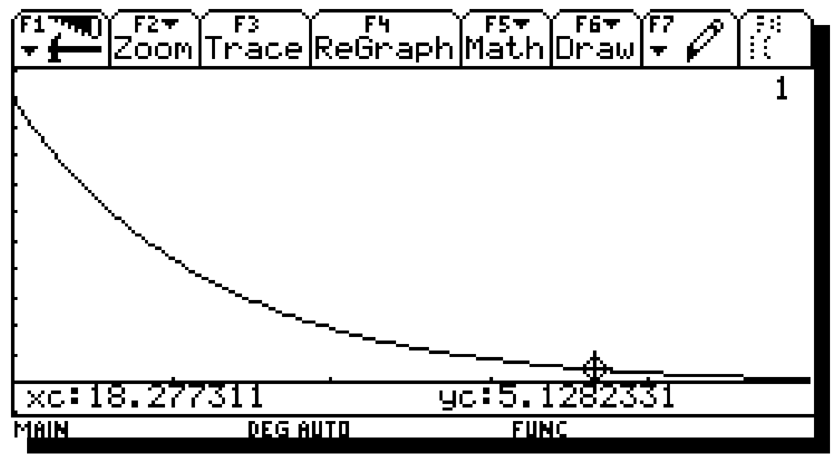
At end of hour	Amount Remaining
1	85.00
2	72.25
3	61.41
4	52.20
5	44.37
6	37.71
7	32.06

- Graph



Solving Exponential Equations Graphically

- For our medication example when does the amount of medication amount to less than 5 mg
- Graph the function for 0 < t < 25
- Use the graph to determine when



$$M(t) = 100 \cdot 0.85^t < 5.0$$

General Formula

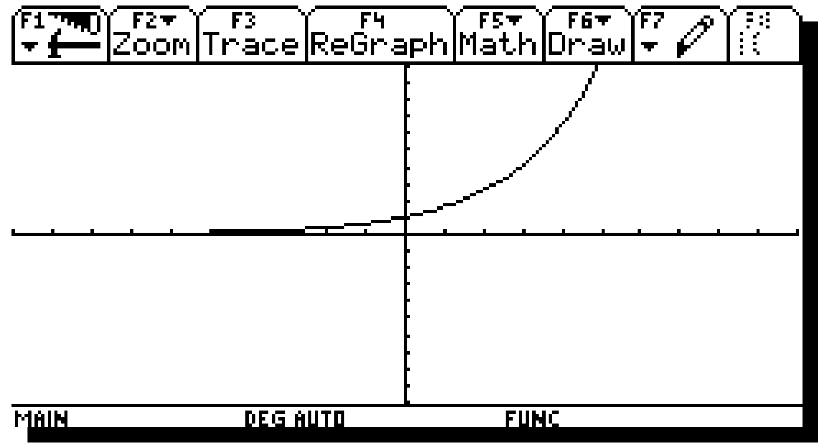
- All exponential functions have the general format:

$$f(t) = A \cdot B^t$$

- Where
  - A = initial value
  - B = growth rate
  - t = number of time periods

Typical Exponential Graphs

- When B > 1



$$f(t) = A \cdot B^t$$

- When B < 1