Math 120 Lessons

[08/29] Algebra Essentials, Numbers- A1 {}

Working with sets {1.1A The real number system p1-2}

Intersection and union of sets

DEFINITION

If A and B are sets, the **intersection** of A with B, denoted $A \cap B$, is the set consisting of elements that belong to both A and B. The **union** of A with B, denoted $A \cup B$, is the set consisting of elements that belong to either A or B, or both.

EXAMPLE 2 **Finding the Intersection and Union of Sets**

Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$. Find:

(a) $A \cap B$ (b) $A \cup B$ (c) $B \cap (A \cup C)$

Solution

(a) $A \cap B = \{1, 3, 5, 8\} \cap \{3, 5, 7\} = \{3, 5\}$

(b) $A \cup B = \{1, 3, 5, 8\} \cup \{3, 5, 7\} = \{1, 3, 5, 7, 8\}$

(c) $B \cap (A \cup C) = \{3, 5, 7\} \cap (\{1, 3, 5, 8\} \cup \{2, 4, 6, 8\})$

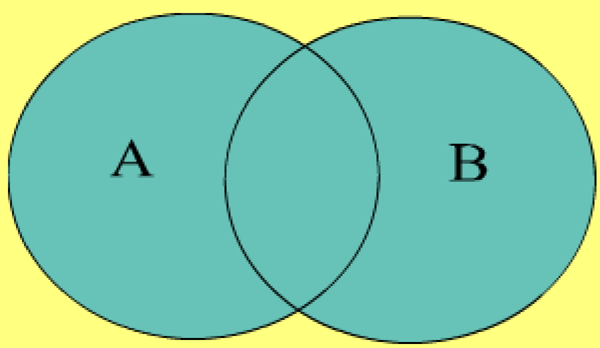
$= \{3, 5, 7\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{3, 5\}$

Now Work PROBLEM 13

Union

In this definition, A and B denote two sets, and are represented in the Venn diagrams by circles. The operations of union and intersection are demonstrated in the diagrams by means of shading.

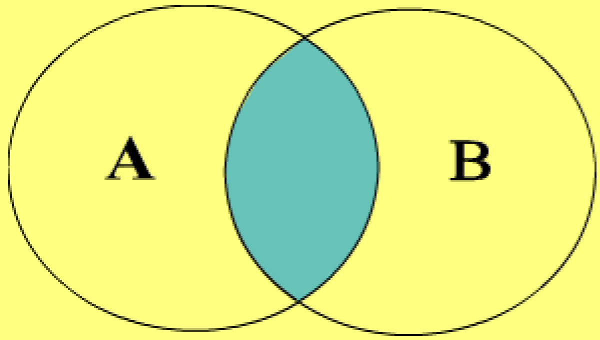
The **union** of A and B, denoted $A \cup B$, is the set $\{ x \mid x \in A \text{ or } x \in B \}$
That is, an element x is in $A \cup B$ if it is in the set A, the set B or both.
Note that the union of A and B contains both individual sets.



Intersection

In this definition, A and B denote two sets, and are represented in the Venn diagrams by circles. The operations of union and intersection are demonstrated in the diagrams by means of shading.

The **intersection** of A and B, denoted $A \cap B$, is the set $\{ x \mid x \in A \text{ and } x \in B \}$
That is, an element x is in $A \cap B$ if it is in both A and B. Note that the intersection of A and B is contained in each individual set.



Example 6: Set Operations

Simplify each of the following set expressions, if possible

a. $[-5, 5] \cup [1, 7)$

Solution:

Picture these intervals on a number line. Because they overlap, their union can be described as a single interval, from the left-most point to the right-most point.

Answer: $[-5, 7)$

Recall, the bracket is a closed interval, meaning that the point is included in the interval. The parenthesis is an open interval, meaning the point itself is not included.

b. $[-5, 5] \cap [1, 7)$

Solution:

We already realized in part a. that these intervals overlap. Their intersection consists of those parts contained in both, or the overlapping section.

Answer: $[1, 5]$

Be sure to carry forward the same notation. Do not replace a closed interval with an open one, or vice versa.

Example 6: Set Operations (cont.)

c. $(-2, 6) \cap (6, 10]$

Solution:

Because these intervals are open at the point 6, they do not overlap. They have no elements in common, and their intersection is the Ø.

Answer: Ø

d. $(-\infty, 10) \cup [7, \infty)$

Solution:

These intervals do overlap. Thus, their union ranges from negative infinity to infinity, and represents the entire set of real numbers.

Answer: $(-\infty, \infty)$

Real Numbers

Types of Real Numbers

The Natural (or Counting) Numbers: This is the set of numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$. The set is infinite, so in list form we can only write the first few such numbers.

The Whole Numbers: This is the set of natural numbers with 0 added: $\{0, 1, 2, 3, 4, 5, \ldots\}$. Again, we can only list the first few members of this set. No special symbol will be assigned to this set in this course.

The Integers: This is the set of natural numbers, their negatives, and 0. As a list, this is the set $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$. Note that the list continues indefinitely in both directions.

The Rational Numbers: This is the set, with symbol $\mathbb{Q}$ for quotient, of *ratios* of integers (hence the name). That is, any rational number can be written in the form $\frac{p}{q}$, where p and q are both integers and $q \neq 0$. When written in decimal form, rational numbers either terminate or repeat a pattern of digits past some point.

The Irrational Numbers: Every real number that is not rational is, by definition, irrational. In decimal form, irrational numbers are non-terminating and non-repeating. No special symbol will be assigned to this set in this course.

The Real Numbers: Every set above is a subset of the set of real numbers, which is denoted $\mathbb{R}$. Every real number is either rational or irrational, and no real number is both.

Example 1 : Identifying Types of Real Numbers

In the following set identify the **a**. natural numbers, **b**. whole numbers, **c**. integers, **d**. rational numbers, and **e**. irrational numbers.

$$S = \{ -\sqrt{4},\ \frac{5}{6},\ 0,\ 2\sqrt{2},\ 5.87,\ 2\sqrt{16},\ 3\pi,\ 8^4 \}$$

Solution:

a. The natural numbers in S are $2\sqrt{16}$ and 8^4. $2\sqrt{16}$ is a natural number since $2\sqrt{16} = 8$.

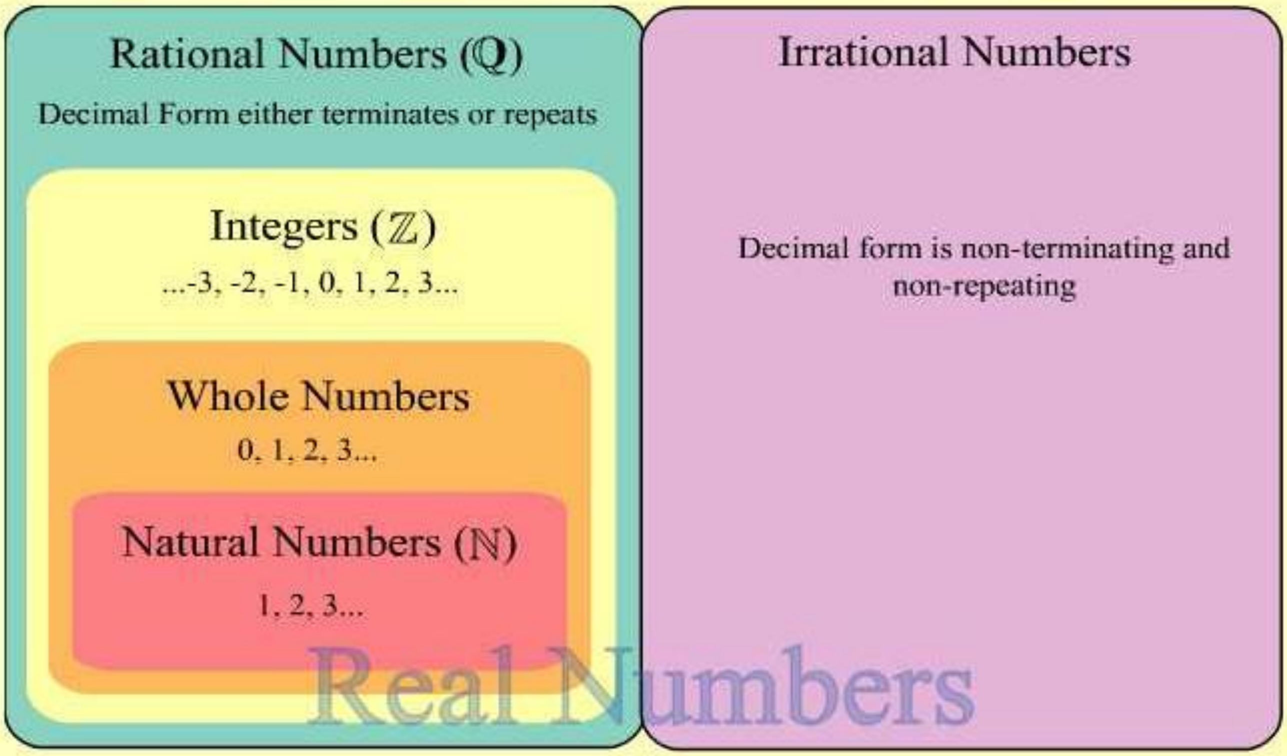
b. The whole numbers in S are 0, $2\sqrt{16}$ and 8^4.

c. The integers in S are $-\sqrt{4}$, 0, $2\sqrt{16}$ and 8^4.

d. The rational numbers are $\frac{5}{6}$, 5.87, $-\sqrt{4}$, 0, $2\sqrt{16}$ and 8^4. Any integer p automatically qualifies as a rational number since it can be written as $\frac{p}{1}$.

e. The only irrational numbers in S are $2\sqrt{2}$ and 3π. Although well known now, the irrationality of $\sqrt{2}$ came as a bit of a surprise to the early Greek mathematicians who discovered this fact, and the irrationality of π was not proven until 1767.

The following figure shows the relationships among the subsets of $\mathbb{R}$ (the real number system) defined earlier. This figure indicates, for example, that every natural number is automatically a whole number, and also an integer, and also a rational number.



Distributive Property

$$a \cdot (b + c) = ab + ac$$

Zero-Product Property

If $ab = 0$, then either $a = 0$ or $b = 0$ or both equal 0.

Zero-Factor Property

Let A and B represent algebraic expressions. If the product of A and B is 0, then at least one of A and B is itself 0. Using the symbol $\Rightarrow$ for "implies", we write

$$AB = 0 \Rightarrow A = 0 \text{ or } B = 0.$$

The Real number line

The Real Number Line

It is often convenient and instructive to depict the set of real numbers as a horizontal line, with each point on the line representing a unique real number and each real number associated with a unique point on the line. The real number corresponding to a given point is called the **coordinate** of that point. Thus, one (and only one) point on the line represents the number 0, and this point is called the **origin** of the real number line. Points to the right of the origin represent positive real numbers, while points to the left of the origin represent negative real numbers.

Figure 2 is an illustration of the real number line with several points plotted. Note that two irrational numbers are plotted, though their locations on the line are necessarily approximations.

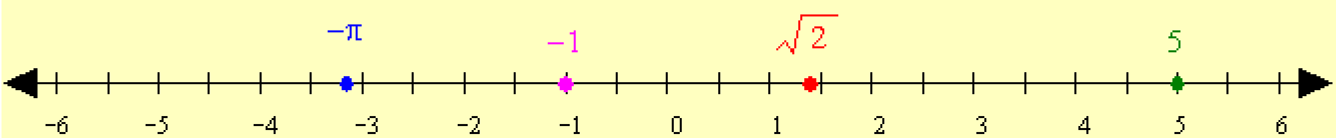


Figure 2: The Real Number Line

Graph inequalities

We choose which portion of the real number line to depict and the physical length that represents one unit as suggested by the numbers that we wish to plot. For example:

Example 2 : Plotting Real Numbers

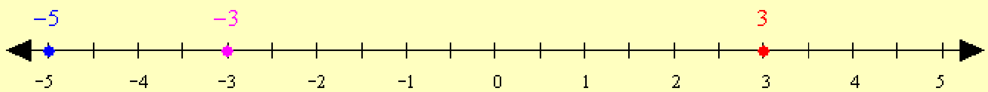
Plot

a. $-5, -3,$ and 3

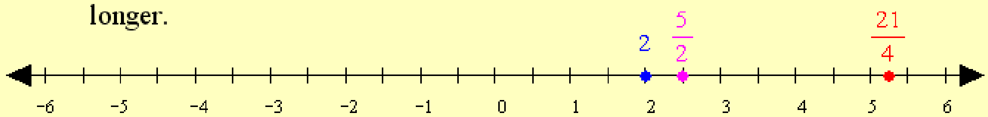
b. $2, \frac{5}{2}$, and $\frac{21}{4}$

Solution:

a. If we want to plot the numbers $-5, -3,$ and 3, we might construct the diagram below.



b. If we want to plot the numbers $2, \frac{5}{2}$, and $\frac{21}{4}$, we might make the unit interval longer.



Inequality Symbols (Order)

Symbol	**Meaning**
$a < b$ (read "a is less than b")	a lies to the left of b on the number line.
$a \le b$ (read "a is less than or equal to b")	a lies to the left of b or is equal to b.
$b > a$ (read "b is greater than a")	b lies to the right of a on the number line.
$b \ge a$ (read "b is greater than or equal to a")	b lies to the right of a or is equal to a.

The two symbols $<$ and $>$ are called *strict* inequalities, while the symbols $\le$ and $\ge$ are *non–strict* inequalities.

Example 3 : Qualifying Inequalities

a. $5 \le 9$, since 5 lies to the left of 9.

b. $5 \le 5$, since 5 is equal to 5. Note that for every real number a, $a \le a$ and $a \ge a$.

c. $-7 > -163$, since -7 lies to the right of -163.

d. The statement "5 is greater than -2" can be written $5 > -2$.

e. The statement "a is less than or equal to $b + c$" can be written $a \le b + c$.

f. The statement "x is strictly less than y" can be written $x < y$.

g. The negation of the statement $a \le b$ is the statement $a > b$.

h. If $a \le b$ and $a \ge b$, then it must be the case that $a = b$.

Notation	**Meaning**
(a, b)	$\{x \mid a < x < b\}$, or all real numbers strictly between a and b.
$[a, b]$	$\{x \mid a \le x \le b\}$, or all real numbers between a and b, including both a and b.
$(a, b]$	$\{x \mid a < x \le b\}$, or all real numbers between a and b, including b but not a.
$(-\infty, b)$	$\{x \mid x < b\}$, or all real numbers less than b.
$[a, \infty)$	$\{x \mid x \ge a\}$, or all real numbers greater than or equal to a.

EXAMPLE 4 **Graphing Inequalities**

(a) On the real number line, graph all numbers x for which $x > 4$.
(b) On the real number line, graph all numbers x for which $x \le 5$.

Solution

(a) See Figure 8. Notice that we use a left parenthesis to indicate that the number 4 is *not* part of the graph.

(b) See Figure 9. Notice that we use a right bracket to indicate that the number 5 *is* part of the graph.

Figure 8

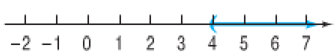
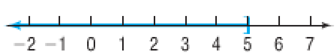


Figure 9



Now Work PROBLEM 41

Find Distance on the Real number line {1.1A p3-4}

Absolute Value

Absolute Value and Distance on the Real Number Line

In addition to order, the depiction of the set of real numbers as a line leads to the notion of *distance*. Physically, distance is a well-understood concept: the distance between two objects is a non-negative number, dependent on a choice of measuring system, indicating how close the objects are to one another. The mathematical idea of **absolute value** gives us a means of defining distance in a mathematical setting.

Absolute Value

The **absolute value** of a real number a, denoted as $|a|$, is defined by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

The absolute value of a number is also referred to as its **magnitude**; it is the non-negative number corresponding to its distance from the origin.

Note that this definition implicitly gives us a system of measurement: 1 and -1 are the two real numbers which have a magnitude of 1, and so the distance between 0 and 1 (or between -1 and 0) is one unit. Note also that 0 is the only real number whose absolute value is 0.

Example 6 : Determining Distance on the Real Number Line

a. $|-\pi| = |\pi| = \pi$. Both $-\pi$ and π are π units from 0.

b. $|17-3| = |3-17| = 14$. 17 and 3 are 14 units apart.

c. $\dfrac{|-7|}{-7} = \dfrac{7}{-7} = -1$.

d. $\dfrac{|7|}{7} = \dfrac{7}{7} = 1$.

e. $-|-5| = -5$. **Note:** The negative sign outside the absolute value symbol is not affected by the absolute value. Compare this with the fact that $-(-5) = 5$.

f. $|\sqrt{7}-2| = \sqrt{7}-2$. Even without a calculator, we know $\sqrt{7}$ is larger than 2 (since $2 = \sqrt{4}$), so $\sqrt{7}-2$ is positive and hence $|\sqrt{7}-2| = \sqrt{7}-2$.

g. $|\sqrt{7}-19| = 19-\sqrt{7}$. In contrast to the last example, we know $\sqrt{7}-19$ is negative, so its absolute value is $19-\sqrt{7}$.

The previous examples illustrate some of the basic properties of absolute value. The list of properties below can all be derived from the definition of absolute value.

Properties of Absolute Values

For all real numbers a and b,

1. $|a| \geq 0$

2. $|-a| = |a|$

3. $a \leq |a|$

4. $|ab| = |a||b|$

5. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, b \neq 0$

6. $|a+b| \leq |a| + |b|$. (This is called the **triangle inequality**, as it is a reflection of the fact that one side of a triangle is never longer than the sum of the other two sides.)

The following example also illustrates some of the properties of absolute values.

Example 7 : Finding Absolute values

a. $|(-3)(5)| = |-15| = 15 = |-3||5|$.

b. $1 = |-3+4| \leq |-3| + |4| = 7$.

c. $7 = |-3-4| \leq |-3| + |-4| = 7$.

d. $\left|\frac{-3}{7}\right| = \frac{|-3|}{|7|} = \frac{3}{7}$.

If P and Q are two points on a real number line with coordinates a and b, respectively, the **distance between P and Q,** denoted by $d(P, Q)$, is

$$d(P, Q) = |b - a|$$

Since $|b - a| = |a - b|$, it follows that $d(P, Q) = d(Q, P)$.

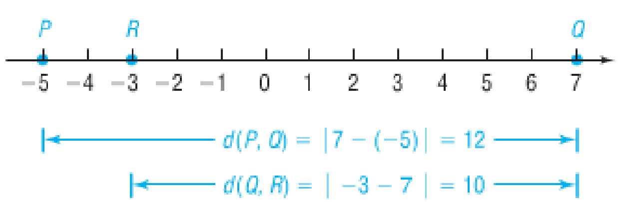
Finding Distance on a Number Line

Let P, Q, and R be points on a real number line with coordinates -5, 7, and -3, respectively. Find the distance

(a) between P and Q (b) between Q and R

See Figure 11.

Figure 11



(a) $d(P, Q) = |7 - (-5)| = |12| = 12$

(b) $d(Q, R) = |-3 - 7| = |-10| = 10$

Now Work PROBLEM 47

Distance on the Real Number Line

Given two real numbers a and b, the **distance** between them is defined to be $|a - b|$. In particular, the distance between a and 0 is $|a - 0|$, or just $|a|$.

Of course, distance should be symmetric. That is, the distance from a to b should be the same as the distance from b to a. Also, no mention was made in the above definition of which of the two numbers a and b is smaller, and our intuition suggests that this is immaterial as far as distance is concerned. These two concerns are really the same, and happily (and not by chance) our mathematical definition of distance coincides with our intuition and is indeed symmetric.

Given two distinct real numbers a and b, exactly one of the two differences $a - b$ and $b - a$ will be negative, and since

$$b - a = -(a - b),$$

the definition of absolute value makes it clear that these two differences have the same magnitude. That is,

$$|a - b| = |b - a|.$$

A few examples should make the above points clear.

Evaluate Algebraic expressions {1.1B Arithmetic of Algebraic Expressions}

Example 1: Components and Terminology

a. Consider the following expression:

$$-27y^5(2x^3 + y) + 3\sqrt{y} - 7(x + y^2)$$

The following table illustrates the elements of this expression and their names.

Algebraic Expressions	$-27y^5(2x^3 + y) + 3\sqrt{y} - 7(x + y^2)$		
Terms	$-27y^5(2x^3 + y)$	$3\sqrt{y}$	$-7(x + y^2)$
Factors	$-27,\ y^5,\ (2x^3 + y)$	$3,\ \sqrt{y}$	$-7,\ (x + y^2)$
Coefficient (of term)	-27	3	-7
Variable Factor (of term)	$y^5(2x^3 + y)$	$\sqrt{y}$	$(x + y^2)$

Note that –27, 3, and –7 are simultaneously **factors** and **coefficients**. Also, both $(2x^3 + y)$ and $(x + y^2)$ are made up of two *terms*, but they are not terms of the *original expression*, just terms of those *factors*.

Example 1: Components and Terminology (cont.)

b. Evaluate the following expression for $x = 4$ and $y = -3$:

$$2x^3 - 3(y - x)$$

Solution:

Replace all x's with 4 's, and replace all y 's with -3 's:

$2x^3 - 3(y - x) = 2(4^3) - 3(-3 - 4)$ Perform the correct calculations.

$= 2(64) - 3(-7)$ Calculate and simplify.

$= 128 + 21$

Answer: $= 149$

Field Properties

In this table, a, b, and c represent arbitrary real numbers. The first five properties apply to addition and multiplication, while the last combines the two.

Name of Property	Additive Version	Multiplicative Version
Closure	$a + b$ is a real number	ab is a real number
Commutative	$a + b = b + a$	$ab = ba$
Associative	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Identity	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Inverse	$a + (-a) = 0$	$a \frac{1}{a} = 1$ (for $a \neq 0$)
Distributive	$a(b + c) = ab + ac$	

Order of Operations

1. If the expression is a fraction, simplify the numerator and denominator individually, according to the guidelines in the following steps.
2. **Parentheses, braces, and brackets** are all used as grouping symbols. Simplify expressions within each set of grouping symbols, if they are present, working from the innermost outward.
3. Simplify all **powers (exponents) and roots**.
4. Perform all **multiplications and divisions** in the expression in the order they occur, working from left to right.
5. Perform all **additions and subtractions** in the expression in the order they occur, working from left to right.

Determine the Domain of a variable

Touch on later

Use the laws of Exponents {1.2a Properties of exponents}

Properties of Exponents

Throughout this table, a and b may be taken to represent constants, variables, or more complicated algebraic expressions. The letters n and m represent integers.

	Property	Example
1.	$a^n \cdot a^m = a^{n+m}$	$(-3)^3 \cdot (-3)^{-1} = (-3)^{3+(-1)} = (-3)^2 = 9$
2.	$\dfrac{a^n}{a^m} = a^{n-m}$	$\dfrac{7^9}{7^{10}} = 7^{9-10} = 7^{-1}$
3.	$a^{-n} = \dfrac{1}{a^n}$	$5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$ and $x^3 = \dfrac{1}{x^{-3}}$
4.	$(a^n)^m = a^{nm}$	$(2^3)^2 = 2^{3\cdot 2} = 2^6 = 64$

Properties of Exponents, cont.

	Property	Example
5.	$(ab)^n = a^n b^n$	$(7x)^3 = 7^3x^3 = 343x^3$ and $(-2x^5)^2 = (-2)^2(x^5)^2 = 4x^{10}$
6.	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{3}{x}\right)^2 = \dfrac{3^2}{x^2} = \dfrac{9}{x^2}$ and $\left(\dfrac{1}{3z}\right)^2 = \dfrac{1^2}{(3z)^2} = \dfrac{1}{9z^2}$
7.	$\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$	$\left(\dfrac{5}{4}\right)^{-3} = \dfrac{4^3}{5^3} = \dfrac{64}{125}$
8.	$\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$	$\dfrac{3^{-2}}{2^{-4}} = \dfrac{2^4}{3^2} = \dfrac{16}{9}$

In the above table, it is assumed that every expression is defined. That is, if an exponent is 0, then the base is non-zero, and if an expression appears in the denominator of a fraction, then that expression is non-zero. Remember that $a^0 = 1$ for every $a \neq 0$.

Simplify the following expressions by using the properties of exponents. Write the final answers with only positive exponents.

a. $(14x^7 - 6x^3 + 9)^0$

b. $\dfrac{(x^4y^2)^{-1}z^{-3}}{x^5z^{-4}}$

c. $\dfrac{(-4x^5y^{-2})^{-3}}{(16x^{-4})^0(xy)^{-2}}$

d. $(9x^2z^{-3})^2(4x^4y)^{-1}$

Incorrect Statements

$x^3 \cdot x^6 = x^{18}$

$2^5 \cdot 2^4 = 4^9$

$(x^3 + 6y)^{-1} = \frac{1}{x^3} + \frac{1}{6y}$

$(5x)^2 = 5x^2$

Corrected Statements

$x^3 \cdot x^6 = x^9$

$2^5 \cdot 2^4 = 2^9$

$(x^3 + 6y)^{-1} = \frac{1}{x^3 + 6y}$

$(5x)^2 = 25x^2$

Evaluate square roots {1.2c Properties of Radicals}

n^{th} Roots and Radical Notation

Case 1: *n* is an even natural number. If a is a non-negative real number and n is an even natural number, $\sqrt[n]{a}$ is the non-negative real number b with the property that $b^n = a$. That is $\sqrt[n]{a} = b \Leftrightarrow a = b^n$. Note that $\left(\sqrt[n]{a}\right)^n = a$ and $\sqrt[n]{a^n} = a$.

Case 2: *n* is an odd natural number. If a is any real number and n is an odd natural number, $\sqrt[n]{a}$ is the real number b (whose sign will be the same as the sign of a) with the property that $b^n = a$. Again, $\sqrt[n]{a} = b \Leftrightarrow a = b^n$, $\left(\sqrt[n]{a}\right)^n = a$ and $\sqrt[n]{a^n} = a$.

The expression $\sqrt[n]{a}$ expresses the n^{th} root of a in **radical notation**. The natural number n is called the **index**, a is the **radicand**, and $\sqrt{}$ is called a **radical sign**. By convention $\sqrt[2]{}$ is usually simply written as $\sqrt{}$.

a. $\sqrt[5]{-243}$

b. $\sqrt{-81}$

c. $-\sqrt{81}$

d. $\sqrt[3]{0} =$

e. $\sqrt[n]{1} =$

$\sqrt[n]{-1} =$

f. $\sqrt[3]{\dfrac{-8}{125}} =$

g. $\sqrt[5]{-\pi^5} =$

h. $\sqrt[4]{(-4)^4}$

Simplified Radical Form

A radical expression is in **simplified form** when:

1. The radicand contains no factor with an exponent greater than or equal to the index of the radical.
2. The radicand contains no fractions.
3. The denominator, if there is one, contains no radical.
4. The greatest common factor of the index and any exponents occurring in the radicand is 1. That is, the index and any exponents in the radicand have no common factor other than 1.

Properties of Radicals

Throughout the following table, a and b may be taken to represent constants, variables, or more complicated algebraic experessions. the letters n and m represent natural numbers. It is assumed that all expressions are defined and are real numbers.

	Property	Example
1.	$\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5,\ \sqrt[7]{3^7} = 3$
2.	$\sqrt[n]{a^n} = \lvert a \rvert$ if n is even	$\sqrt[4]{(-6)^4} = \lvert -6 \rvert = 6$
3.	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt[3]{3x^6y^2} = \sqrt[3]{3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{y^2} = \sqrt[3]{3} \cdot x \cdot x \cdot \sqrt[3]{y^2} = x^2 \cdot \sqrt[3]{3y^2}$
4.	$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{x^4}{16}} = \dfrac{\sqrt[4]{x^4}}{\sqrt[4]{16}} = \dfrac{\lvert x \rvert}{2}$
5.	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{64}} = \sqrt[3]{\sqrt[2]{64}} = \sqrt[6]{64} = 2$

Simplify the following radical expressions:

a. $\sqrt[3]{-24\,a^5b^3}$ **b.** $\sqrt[3]{108\,x\,y^6}$ **c.** $\sqrt{\dfrac{5}{4\,x^2}}$

Solution:

Rationalizing the denominator using a Conjugate

Enter the factor that will rationalize the denominator for the expressions below.

a. $\dfrac{2+\sqrt{10}}{6-\sqrt{2}}$ conjugate =

Simplify the following radical expressions:

a. $\dfrac{3}{\sqrt[3]{32x}}$ **b.** $\dfrac{y}{\sqrt{x}+\sqrt{2y}}$

Solution:

As an aside, there are occasions when rationalizing the numerator is desirable. For instance, some problems in Calculus (which the author encourages *all* college students to take as a consciousness-raising experience!) are much easier to solve after rationalizing the numerator of a given fraction. This is accomplished by the same method, as seen in the next example.

Rationalize the numerator of the fraction $\frac{\sqrt{7}-\sqrt{2}}{1}$.

[08/29] Polynomials Factoring- A3 {1.3 Polynomials and Factoring}

Recognize Monomials

1 Recognize Monomials

DEFINITION

NT The nonnegative integers integers 0, 1, 2, 3, … .

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

$$ax^k$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is the **degree** of the monomial.

EXAMPLE 1 **Examples of Monomials**

Monomial	Coefficient	Degree	
(a) $6x^2$	6	2	
(b) $-\sqrt{2}x^3$	$-\sqrt{2}$	3	
(c) 3	3	0	Since $3 = 3 \cdot 1 = 3x^0, x \neq 0$
(d) $-5x$	-5	1	Since $-5x = -5x^1$
(e) x^4	1	4	Since $x^4 = 1 \cdot x^4$

Now let's look at some expressions that are not monomials.

EXAMPLE 2 **Examples of Nonmonomial Expressions**

(a) $3x^{1/2}$ is not a monomial, since the exponent of the variable x is $\frac{1}{2}$ and $\frac{1}{2}$ is not a nonnegative integer.

(b) $4x^{-3}$ is not a monomial, since the exponent of the variable x is -3 and -3 is not a nonnegative integer.

Now Work PROBLEM 11

Recognize Polynomials

DEFINITION

In Words
A polynomial is a sum of monomials.

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is the **leading coefficient,** and n is the **degree** of the polynomial.

The monomials that make up a polynomial are called its **terms.** If all the coefficients are 0, the polynomial is called the **zero polynomial,** which has no degree.

Polynomials are usually written in **standard form,** beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree. If a power of x is missing, it is because its coefficient is zero.

EXAMPLE 3 **Examples of Polynomials**

Polynomial	Coefficients	Degree
$-8x^3 + 4x^2 + 6x + 2$	$-8, 4, 6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x + (-5)$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 + (-2)x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree

Add or subtract the polynomials, as indicated.

a. $(7x^2 - 2xy + 3y^2) - (-3x^2 - 2xy + 5y^2)$

b. $(12ab^3c + 8b^5c^2 - 3ac^3 - 10) + (4 + 5ac^3 - ab^3c)$

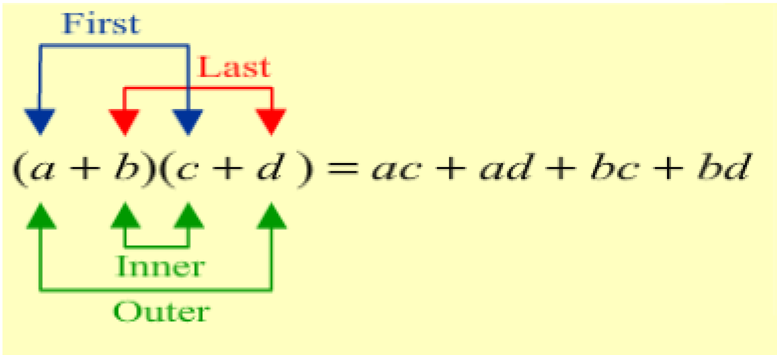
Ans

The first step is to identify like terms, and group these together. Note: remember to distribute the minus sign over all the terms in the second polynomial. The like terms are then combined by using the distributive property.

Ans

Know formulas for Special Products

FOIL



Difference of Two Squares

$$(x - a)(x + a) = x^2 - a^2 \qquad \textbf{(2)}$$

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2 \qquad \textbf{(3a)}$$
$$(x - a)^2 = x^2 - 2ax + a^2 \qquad \textbf{(3b)}$$

Cubes of Binomials, or Perfect Cubes

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \qquad \textbf{(4a)}$$
$$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \qquad \textbf{(4b)}$$

Difference of Two Cubes

$$(x - a)(x^2 + ax + a^2) = x^3 - a^3 \qquad \textbf{(5)}$$

Sum of Two Cubes

$$(x + a)(x^2 - ax + a^2) = x^3 + a^3 \qquad \textbf{(6)}$$

Multiply the polynomials, as indicated.

a. $(4ac - 6ab)(3ac + 2ab^3 - abc)$ **b.** $(xy + 2y)(2xy + 5y)$

Special Product Formulas

1. $(A - B)(A + B) = A^2 - B^2$
2. $(A + B)^2 = A^2 + 2AB + B^2$
3. $(A - B)^2 = A^2 - 2AB + B^2$
4. $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
5. $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

Divide Polynomials Using Long Division

EXAMPLE 5 **Dividing Two Integers**

Divide 842 by 15.

Solution

	56	← Quotient
Divisor →	15)842	← Dividend
	75	← $5 \cdot 15$ (subtract)
	92	
	90	← $6 \cdot 15$ (subtract)
	2	← Remainder

So, $\dfrac{842}{15} = 56 + \dfrac{2}{15}$.

In the long division process detailed in Example 5, the number 15 is called the **divisor,** the number 842 is called the **dividend,** the number 56 is called the **quotient,** and the number 2 is called the **remainder.**

To check the answer obtained in a division problem, multiply the quotient by the divisor and add the remainder. The answer should be the dividend.

(Quotient)(Divisor) + Remainder = Dividend

For example, we can check the results obtained in Example 5 as follows:

$$(56)(15) + 2 = 840 + 2 = 842$$

To divide two polynomials, we first must write each polynomial in standard form. The process then follows a pattern similar to that of Example 5. The next example illustrates the procedure.

EXAMPLE 6 **Dividing Two Polynomials**

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \quad \text{is divided by} \quad x^2 + 1$$

Solution Each polynomial is in standard form. The dividend is $3x^3 + 4x^2 + x + 7$, and the divisor is $x^2 + 1$.

REMEMBER A polynomial is in standard form when its terms are written according to descending degrees. ■

STEP 1: Divide the leading term of the dividend, $3x^3$, by the leading term of the divisor, x^2. Enter the result, $3x$, over the term $3x^3$, as follows:

$$\begin{array}{r} 3x \phantom{{}+4x^2+x+7} \\ x^2+1 \overline{)\, 3x^3 + 4x^2 + x + 7} \end{array}$$

STEP 2: Multiply $3x$ by $x^2 + 1$ and enter the result below the dividend.

$$\begin{array}{r} 3x \phantom{{}+4x^2+x+7} \\ x^2+1 \overline{)\, 3x^3 + 4x^2 + x + 7} \\ \underline{3x^3 \phantom{{}+4x^2} + 3x \phantom{{}+7}} \end{array} \quad \leftarrow 3x \cdot (x^2 + 1) = 3x^3 + 3x$$

Notice that we align the $3x$ term under the x to make the next step easier.

STEP 3: Subtract and bring down the remaining terms.

$$\begin{array}{r} 3x \phantom{{}+4x^2+x+7} \\ x^2+1 \overline{)\, 3x^3 + 4x^2 + x + 7} \\ \underline{3x^3 \phantom{{}+4x^2} + 3x \phantom{{}+7}} \\ 4x^2 - 2x + 7 \end{array} \quad \begin{array}{l} \leftarrow \text{Subtract (change the signs and add).} \\ \leftarrow \text{Bring down the } 4x^2 \text{ and the 7.} \end{array}$$

STEP 4: Repeat Steps 1–3 using $4x^2 - 2x + 7$ as the dividend.

$$\begin{array}{r} 3x + 4 \phantom{{}+x+7} \\ x^2+1 \overline{)\, 3x^3 + 4x^2 + x + 7} \\ \underline{3x^3 \phantom{{}+4x^2} + 3x \phantom{{}+7}} \\ 4x^2 - 2x + 7 \\ \underline{4x^2 \phantom{{}-2x} + 4} \\ -2x + 3 \end{array} \quad \begin{array}{l} \leftarrow \text{Divide } 4x^2 \text{ by } x^2 \text{ to get 4.} \\ \leftarrow \text{Multiply } x^2 + 1 \text{ by 4; subtract.} \end{array}$$

Since x^2 does not divide $-2x$ evenly (that is, the result is not a monomial), the process ends. The quotient is $3x + 4$, and the remainder is $-2x + 3$.

✓**Check:** (Quotient)(Divisor) + Remainder

$$= (3x + 4)(x^2 + 1) + (-2x + 3)$$
$$= 3x^3 + 3x + 4x^2 + 4 + (-2x + 3)$$
$$= 3x^3 + 4x^2 + x + 7 = \text{Dividend}$$

Then

$$\frac{3x^3 + 4x^2 + x + 7}{x^2 + 1} = 3x + 4 + \frac{-2x + 3}{x^2 + 1}$$

Factor Polynomials

Identifying Common Monomial Factors

Polynomial	Common Monomial Factor	Remaining Factor	Factored Form
$2x + 4$	2	$x + 2$	$2x + 4 = 2(x + 2)$
$3x - 6$	3	$x - 2$	$3x - 6 = 3(x - 2)$
$2x^2 - 4x + 8$	2	$x^2 - 2x + 4$	$2x^2 - 4x + 8 = 2(x^2 - 2x + 4)$
$8x - 12$	4	$2x - 3$	$8x - 12 = 4(2x - 3)$
$x^2 + x$	x	$x + 1$	$x^2 + x = x(x + 1)$
$x^3 - 3x^2$	x^2	$x - 3$	$x^3 - 3x^2 = x^2(x - 3)$
$6x^2 + 9x$	$3x$	$2x + 3$	$6x^2 + 9x = 3x(2x + 3)$

Notice that, once all common monomial factors have been removed from a polynomial, the remaining factor is either a prime polynomial of degree 1 or a polynomial of degree 2 or higher. (Do you see why?)

The list of special products (2) through (6) given earlier provides a list of factoring formulas when the equations are read from right to left. For example, equation (2) states that if the polynomial is the difference of two squares, $x^2 - a^2$, it can be factored into $(x - a)(x + a)$. The following example illustrates several factoring techniques.

Method 1: Greatest Common Factor. Factoring out those factors common to all the terms in an expression is the easiest factoring method to apply, and should be done first if possible. The greatest common factor (GCF) among all the terms is simply the product of all the factors common to each. For instance, $2x$ is a factor common to all the terms in the polynomial $12x^5 - 4x^2 + 8x^3z^3$ but is not the greatest common factor. The Greatest Common Factor method is a matter of applying the distributive property to "un-distribute" the greatest common factor.

Use the Greatest Common Factor method to factor the following polynomials.

a. $8a^2b - 4ab + 6a^3b^3$ **b.** $-30x^2y + 5y$ **c.** $4\left(x^2 + xy\right) + 3\left(x^2 + xy\right)$

Method 2: Factoring by Grouping. Many polynomials have a GCF of 1, and the first factoring method is therefore not directly applicable. But if the terms of the polynomial are grouped in a suitable way, the GCF method may apply to each group, and a common factor might subsequently be found among the groups. Factoring by Grouping is the name given to this process, and it is important to realize that this is a trial and error process. Your first attempt at grouping and factoring may not succeed, and you may have to try several different ways of grouping the terms.

Use the Factor by Grouping method to factor the following polynomials.

a. $yz - 2yz^2 + 12yz^3 - 6yz^2$ **b.** $3ab^2 - 4a - 2a^2b + 6b$

Caution!

One common error in factoring is to stop after groups within the original polynomial have been factored. For instance, while we have done some factoring to achieve the expression $yz(1-6z)-2yz^2(1-6z)$ in Example 5a, this is *not* in factored form. An expression is only factored if it is written as a *product* of two or more factors. The expression $yz(1-6z)-2yz^2(1-6z)$ is a sum of two smaller expressions.

Method 3: Factoring Special Binomials. Three types of binomials can always be factored by following the patterns outlined below. You should verify the truth of the patterns by multiplying out the products on the right-hand side of each one.

In the following, A and B represent algebraic expressions.

Difference of Two Squares: $A^2-B^2=(A-B)(A+B)$.

Difference of Two Cubes: $A^3-B^3=(A-B)\left(A^2+AB+B^2\right)$.

Sum of Two Cubes: $A^3+B^3=(A+B)\left(A^2-AB+B^2\right)$.

Factor the following binomials

a. $25a^4-4b^6$ **b.** $512x^9y^6z^3-8x^6y^3$

Method 4: Factoring Trinomials. In factoring a trinomial of the form ax^2+bx+c, the goal is to find two binomials $px+q$ and $rx+s$ such that

$$ax^2+bx+c=(px+q)(rx+s).$$

Since $(px+q)(rx+s)=prx^2+(ps+qr)x+qs$, we seek p, q, r and s such that $a=pr$, $b=ps+qr$ and $c=qs$:

$$ax^2+bx+c=\underbrace{pr}_{a}x^2+\underbrace{(ps+qr)}_{b}x+\underbrace{qs}_{c}$$

In general, this may require much trial and error, but the following guidelines will help.

Complete the square

Completing the Square

Identify the coefficient of the first-degree term. Multiply this coefficient by $\frac{1}{2}$ and then square the result. That is, determine the value of b in $x^2 + bx$ and compute $\left(\frac{1}{2}b\right)^2$.

Completing the Square

Determine the number that must be added to each expression to complete the square. Then factor the expression.

Start	Add	Result	Factored Form
$y^2 + 8y$	$\left(\frac{1}{2}\cdot 8\right)^2 = 16$	$y^2 + 8y + 16$	$(y + 4)^2$
$x^2 + 12x$	$\left(\frac{1}{2}\cdot 12\right)^2 = 36$	$x^2 + 12x + 36$	$(x + 6)^2$
$a^2 - 20a$	$\left(\frac{1}{2}\cdot (-20)\right)^2 = 100$	$a^2 - 20a + 100$	$(a - 10)^2$
$p^2 - 5p$	$\left(\frac{1}{2}\cdot (-5)\right)^2 = \frac{25}{4}$	$p^2 - 5p + \frac{25}{4}$	$\left(p - \frac{5}{2}\right)^2$

[08/31] Rational Expressions-A5{ 1.8a Rational expressions and equations}

Rational Expressions

A **rational expression** is an expression that can be written as a *ratio* of two polynomials $\frac{P}{Q}$. Of course, such a fraction is undefined for any value(s) of the variable(s) for which $Q = 0$. A given rational expression is *simplified* or *reduced* when P and Q contain no common factors (other than 1 and -1).

To simplify rational expressions, we factor the polynomials in the numerator and denominator completely and then cancel any common factors. It is important to remember, however, that the simplified rational expression may be defined for values of the variable (or variables) that the original (unsimplified) expression is not, and the two versions are equal only where they are both defined. That is, if A, B and C represent algebraic expressions,

$$\frac{AC}{BC} = \frac{A}{B} \text{ only where } B \neq 0 \text{ and } C \neq 0.$$

Reduce a Rational Expression to Lowest Terms

a. $\dfrac{x^3 + 27}{x^2 + 3x}$ **b.** $\dfrac{x^2 + 7x - 18}{2 - x}$

Caution!

Remember that only common *factors* can be cancelled! A very common error is to think that common terms from the numerator and denominator can be cancelled. For instance, the statement $\dfrac{\not{x} + 4}{x^{\not{2}}} = \dfrac{4}{x}$ is incorrect. It is not possible to factor $x + 4$ at all, and the x that appears in the numerator is *not* a factor that can be cancelled with one of the x's in the denominator. The expression $\dfrac{x+4}{x^2}$ is already simplified as far as possible.

Multiply and Divide Rational Expressions

Multiply or divide the following rational expressions, as indicated.

a. $\dfrac{x^2 + 3x - 28}{x + 3} \cdot \dfrac{x - 5}{x^2 + 8x + 7}$ **b.** $\dfrac{x^2 + x - 20}{4x} \div \dfrac{x^2 - 8x + 16}{16x^3}$

A **complex rational expression** is a fraction in which the numerator or denominator (or both) contains at least one rational expression. Complex rational expressions can always be rewritten as simple rational expressions. One way to do this is to simplify the numerator and denominator individually and then divide the numerator by the denominator as in Example 3b. Another way, which is frequently faster, is to multiply the numerator and denominator by the LCD of all the fractions that make up the complex rational expression. This method will be illustrated in the next two examples.

Add and Subtract Rational Expressions

Add or subtract the following rational expressions, as indicated.

a. $\dfrac{3x - 2}{x^2 - 6x + 8} - \dfrac{2x}{x^2 - 16}$ **b.** $\dfrac{x + 4}{x + 5} + \dfrac{x^2 - x - 6}{x^2 - 3x - 10} - \dfrac{x^2 + 3x - 20}{x^2 - 25}$

Use the least common Multiple Method

Use the Least Common Multiple Method

If the denominators of two rational expressions to be added (or subtracted) have common factors, we usually do not use the general rules given by equations (5a) and (5b). Just as with fractions, we apply the **least common multiple (LCM) method.** The LCM method uses the polynomial of least degree that has each denominator polynomial as a factor.

The LCM Method for Adding or Subtracting Rational Expressions

The Least Common Multiple (LCM) Method requires four steps:

STEP 1: Factor completely the polynomial in the denominator of each rational expression.

STEP 2: The LCM of the denominator is the product of each of these factors raised to a power equal to the greatest number of times that the factor occurs in the polynomials.

STEP 3: Write each rational expression using the LCM as the common denominator.

STEP 4: Add or subtract the rational expressions using equation (4).

Simplify Complex Rational expressions

Simplify the following complex rational expressions.

a. $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$ **b.** $\dfrac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}$

[08/31] Solving Linear and Quadratic Equations –A6
{1.5a linear equations in one variable, 1.7a Quadratic equations in one variable}

Linear Equations in One Variable

A **linear equation in one variable**, say the variable x, is an equation that can be transformed into the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$. Such equations are also called **first-degree** equations, as x appears to the first power.

a. $5(x-2)+3x = 1-4\left(x+\dfrac{3}{4}\right)$ **b.** $5x-8=5(x-2)$

Solve Equations by Factoring

Zero-Factor Property

Let A and B represent algebraic expressions. If the product of A and B is 0, then at least one of A and B is itself 0. That is,

$$AB = 0 \Rightarrow A = 0 \text{ or } B = 0$$

$$
\begin{aligned}
x^3 &= 4x \\
x^3 - 4x &= 0 \\
x(x^2 - 4) &= 0 && \text{Factor.} \\
x(x-2)(x+2) &= 0 && \text{Factor again.} \\
x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x + 2 &= 0 && \text{Apply the Zero-Product Property.} \\
x = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x &= -2 && \text{Solve for } x.
\end{aligned}
$$

The solution set is $\{-2, 0, 2\}$.

Check:

$x = -2$: $(-2)^3 = -8$ and $4(-2) = -8$ — -2 is a solution.

$x = 0$: $0^3 = 0$ and $4 \cdot 0 = 0$ — 0 is a solution.

$x = 2$: $2^3 = 8$ and $4 \cdot 2 = 8$ — 4 is a solution.

(b) Group the terms of $x^3 - x^2 - 4x + 4 = 0$ as follows:

$$(x^3 - x^2) - (4x - 4) = 0$$

Factor out x^2 from the first grouping and 4 from the second.

$$x^2(x - 1) - 4(x - 1) = 0$$

This reveals the common factor $(x - 1)$, so we have

$$
\begin{aligned}
(x^2 - 4)(x - 1) &= 0 \\
(x - 2)(x + 2)(x - 1) &= 0 && \text{Factor again.} \\
x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad x - 1 &= 0 && \text{Apply the Zero-Product Property.} \\
x = 2 \qquad x = -2 \qquad x &= 1 && \text{Solve for } x.
\end{aligned}
$$

The solution set is $\{-2, 1, 2\}$.

Solve Equations Involving Absolute Value

Solving an Equation Involving Absolute Value

Solve the equation: $|x + 4| = 13$

There are two possibilities.

$$
\begin{aligned}
x + 4 = 13 \quad &\text{or} \quad x + 4 = -13 \\
x = 9 \quad &\text{or} \quad x = -17
\end{aligned}
$$

The solution set is $\{-17, 9\}$.

Solve a Quadratic Equation by Factoring

Quadratic Equations

A **quadratic equation in one variable**, say the variable x, is an equation that can be transformed into the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. Such equations are also called **second-degree** equations, as x appears to the second power. The name ***quadratic*** comes from the Latin word *quadrus*, meaning "square".

Solve the following quadratic equations by factoring.

a. $x^2 + \frac{11x}{3} = \frac{4}{3}$ **b.** $s^2 + 25 = 10s$ **c.** $4x^2 + 28x = 0$

Solve the following quadratic equations by taking square roots.

$$\text{If } x^2 = p \text{ and } p \geq 0, \text{ then } x = \sqrt{p} \text{ or } x = -\sqrt{p}. \qquad \textbf{(3)}$$

a. $(2x + 5)^2 = 12$

Solve a Quadratic Equation by Completing the Square [find better examples***]

Solving a Quadratic Equation by Completing the Square

Solve by completing the square: $2x^2 - 8x - 5 = 0$

First, rewrite the equation as follows:

$$2x^2 - 8x - 5 = 0$$
$$2x^2 - 8x = 5$$

Next, divide both sides by 2 so that the coefficient of x^2 is 1. (This enables us to complete the square at the next step.)

$$x^2 - 4x = \frac{5}{2}$$

Finally, complete the square by adding $\left[\frac{1}{2}(-4)\right]^2 = 4$ to both sides.

$$x^2 - 4x + 4 = \frac{5}{2} + 4$$
$$(x - 2)^2 = \frac{13}{2}$$
$$x - 2 = \pm\sqrt{\frac{13}{2}} \quad \text{Use the Square Root Method.}$$
$$x - 2 = \pm\frac{\sqrt{26}}{2} \quad \sqrt{\frac{13}{2}} = \frac{\sqrt{13}}{\sqrt{2}} = \frac{\sqrt{13}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{26}}{2}$$
$$x = 2 \pm \frac{\sqrt{26}}{2}$$

The solution set is $\left\{2 - \frac{\sqrt{26}}{2}, 2 + \frac{\sqrt{26}}{2}\right\}$

Solve a Quadratic Equation Using the Quadratic Formula

The Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Note that the equation has a single solution if $b^2 - 4ac = 0$ and two complex solutions that are conjugate of one another if $b^2 - 4ac < 0$.

Solving a Quadratic Equation Using the Quadratic Formula

Use the quadratic formula to find the real solutions, if any, of the equation

$$3x^2 - 5x + 1 = 0$$

The equation is in standard form, so we compare it to $ax^2 + bx + c = 0$ to find a, b, and c.

$$3x^2 - 5x + 1 = 0$$
$$ax^2 + bx + c = 0 \quad a = 3, b = -5, c = 1$$

With $a = 3$, $b = -5$, and $c = 1$, evaluate the discriminant $b^2 - 4ac$.

$$b^2 - 4ac = (-5)^2 - 4(3)(1) = 25 - 12 = 13$$

Since $b^2 - 4ac > 0$, there are two real solutions, which can be found using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{13}}{2(3)} = \frac{5 \pm \sqrt{13}}{6}$$

The solution set is $\left\{\dfrac{5 - \sqrt{13}}{6}, \dfrac{5 + \sqrt{13}}{6}\right\}$.

Solve the following quadratic equations by using the quadratic formula.

a. $6y^2 - 5y = 1$

[09/07]Complex Numbers-A7{1.4 Complex number system}

The Imaginary Unit *i*

The imaginary unit i is defined as $i = \sqrt{-1}$. In other words, i has the property that its square is -1: $i^2 = -1$.

Square Roots of Negative Numbers

If a is a positive real number, $\sqrt{-a} = i\sqrt{a}$. Note that by this definition, and by a logical extension of exponentiation, $\left(i\sqrt{a}\right)^2 = i^2\left(\sqrt{a}\right)^2 = -a$.

a. $\sqrt{-25} = i\sqrt{25} = i(5) = 5i$. As is customary, we write a constant such as 5 before letters in algebraic expressions, even if, as in this case, the letter is not a variable. Remember that i has a fixed meaning: i is the square root of -1.

b. $\sqrt{-12} = i\sqrt{12} = i\left(2\sqrt{3}\right) = 2i\sqrt{3}$. As is customary, again we write the radical factor last. You should verify that $\left(2i\sqrt{3}\right)^2$ is indeed -12.

c. $i^7 = i^2\, i^2\, i^2\, i = (-1)(-1)(-1)(i) = -i$, and
$i^8 = i^2\, i^2\, i^2\, i^2 = (-1)(-1)(-1)(-1) = 1$. The simple fact that $i^2 = -1$ allows us, by our extension of exponentiation, to determine i^n for any natural number n.

d. $(-i)^2 = (-1)^2\, i^2 = i^2 = -1$. This observation shows that $-i$ also has the property that its square is -1.

Add, subtract, multiply and divide complex numbers

Adding and Subtracting Complex Numbers

(a) $(3 + 5i) + (-2 + 3i) = [3 + (-2)] + (5 + 3)i = 1 + 8i$

(b) $(6 + 4i) - (3 + 6i) = (6 - 3) + (4 - 6)i = 3 + (-2)i = 3 - 2i$

Multiplying Complex Numbers

$$(5 + 3i) \cdot (2 + 7i) = 5 \cdot (2 + 7i) + 3i(2 + 7i) = 10 + 35i + 6i + 21i^2$$

↑ Distributive Property ↑ Distributive Property

$$= 10 + 41i + 21(-1)$$

↑ $i^2 = -1$

$$= -11 + 41i$$

Based on the procedure of Example 2, we define the **product** of two complex numbers as follows:

Simplify the following complex number expressions.

a. $(5 + 6i) + (-7 + 3i)$ **b.** $(-3 + 3i) - (-5 + 3i)$

c. $(6 + 3i)(-5 + 4i)$ **d.** $(7 - 2i)^2$

Simplify the following quotients.

a. $\dfrac{3 + 4i}{4 - i}$ **b.** $(5 - 2i)^{-1}$ **c.** $\dfrac{1}{i}$

Solve quadratic equations in the complex number system.

b. $t^2 + 8t + 20 = 0$