4.1 Intro to polynomials and their graphs

Identify a polynomial.

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (1)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and n is a nonnegative integer. The domain of a polynomial function is the set of all real numbers.

Power Functions

A **power function of degree n** is a monomial function of the form

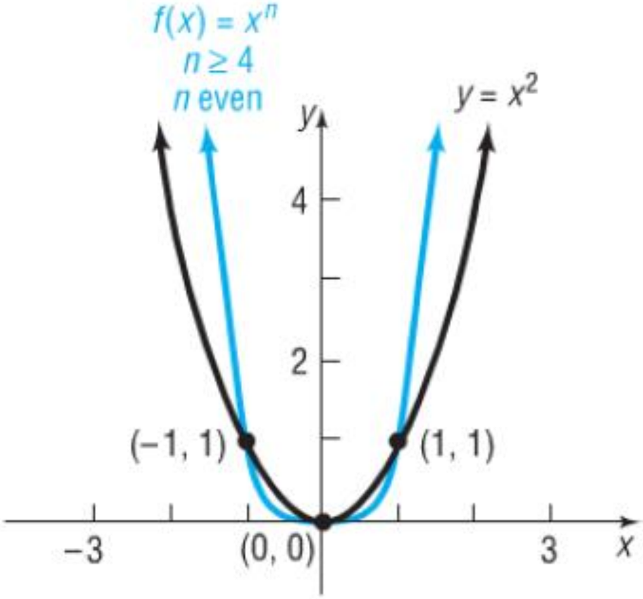
$$f(x) = ax^n \qquad (2)$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

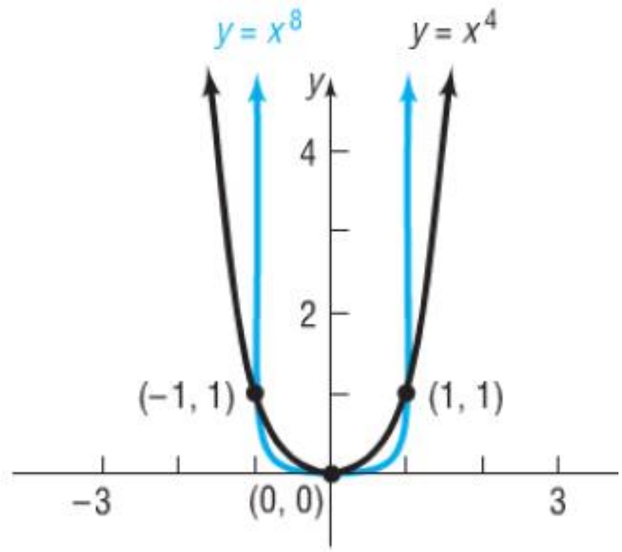
Examples of power functions are

$f(x) = 3x$	$f(x) = -5x^2$	$f(x) = 8x^3$	$f(x) = -5x^4$
degree 1	degree 2	degree 3	degree 4

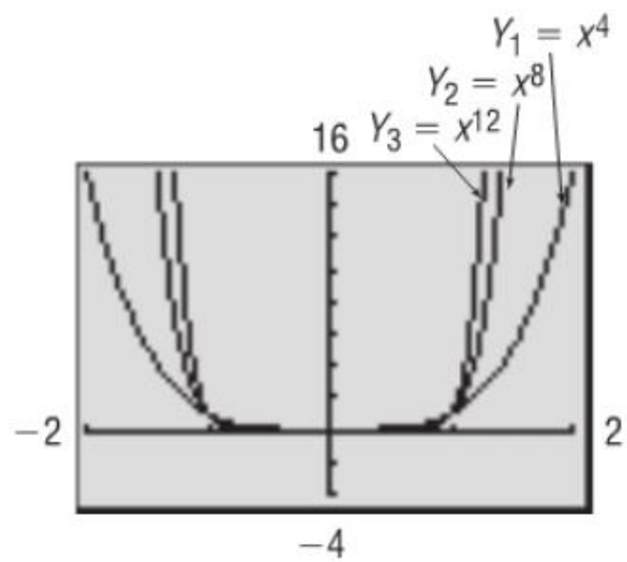
Properties of Power Functions f(x) = x^n, n is a positive even & odd integers



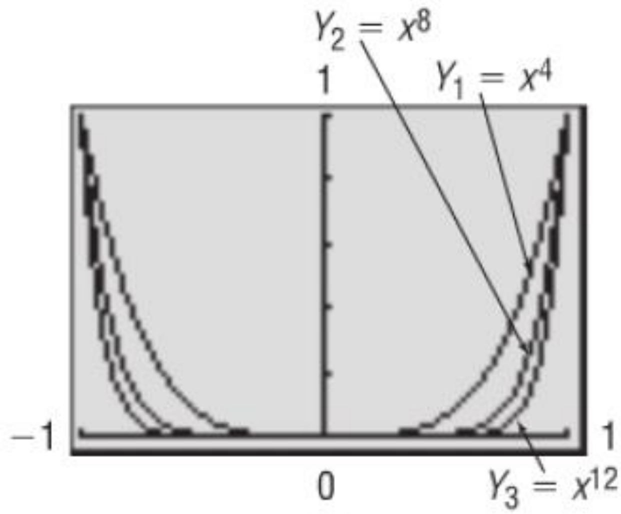
(a)



(b)



(a)

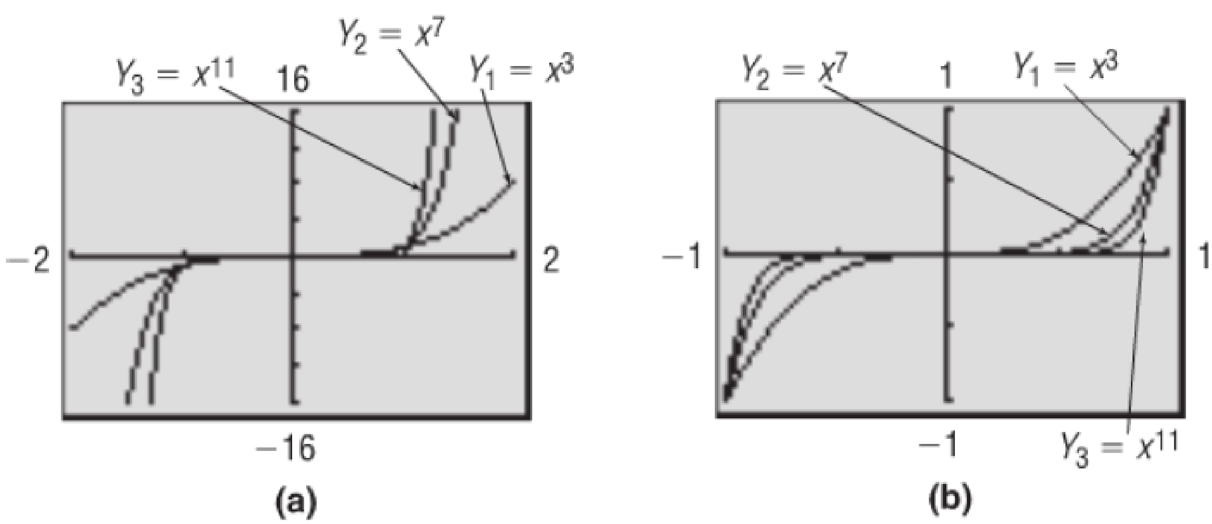


(b)

Properties of Power Functions, $f(x) = x^n$, n Is a Positive Even Integer

1. f is an even function, so its graph is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$.
4. As the exponent n increases in magnitude, the function increases more rapidly when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

ODD INTEGER POWER FUNCTIONS



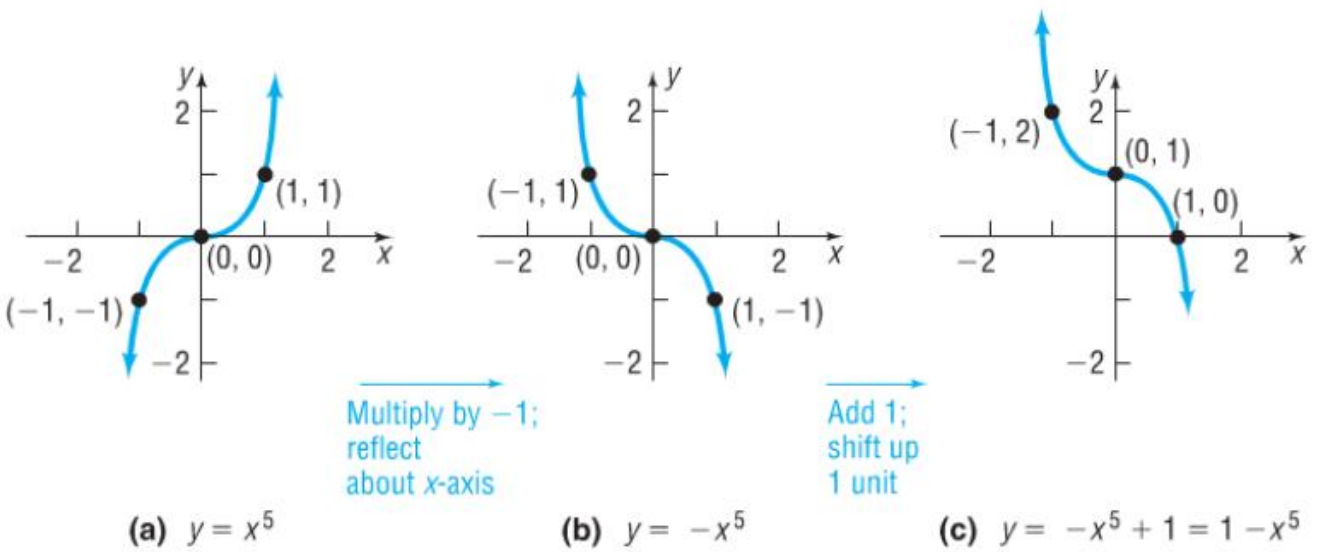
Properties of Power Functions, $f(x) = x^n$, n Is a Positive Odd Integer

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.
4. As the exponent n increases in magnitude, the function increases more rapidly when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Polynomial functions and using transformations.

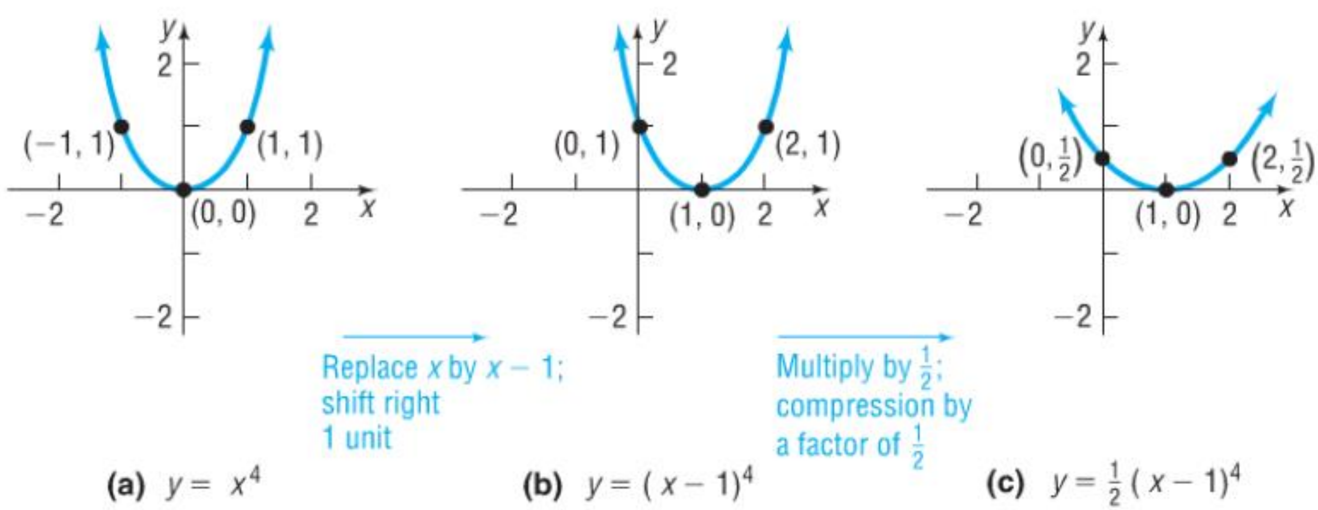
Graph: $f(x) = 1 - x^5$

It is helpful to rewrite f as $f(x) = -x^5 + 1$. Figure 7 shows the required stages.

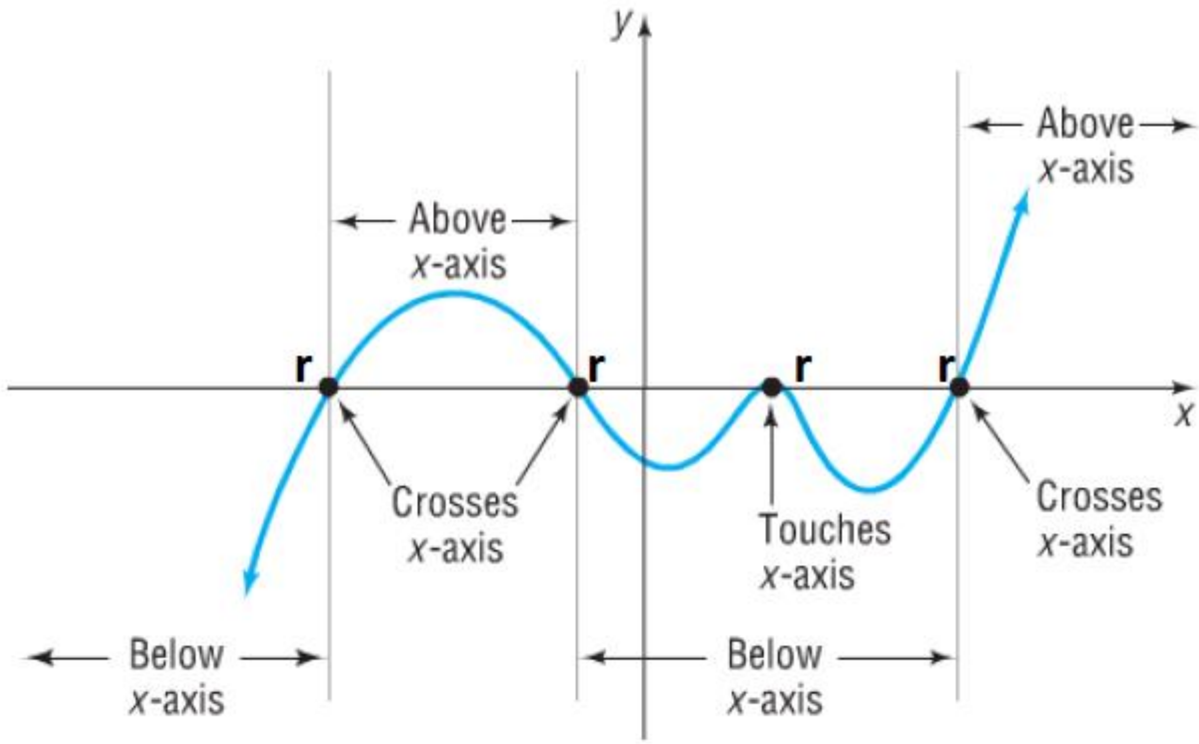


Graph: $f(x) = \frac{1}{2}(x - 1)^4$

Figure 8 shows the required stages.



Identify Real Zeros of a polynomial function and their multiplicity



Polynomials are continuous no gaps or holes

If a polynomial function f is factored completely, it is easy to locate the x-intercepts of the graph by solving the equation $f(x) = 0$ and using the Zero-Product Property. For example, if $f(x) = (x - 1)^2(x + 3)$, then the solutions of the equation

$$f(x) = (x - 1)^2(x + 3) = 0$$

are identified as 1 and -3. That is, $f(1) = 0$ and $f(-3) = 0$.

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f.

As a consequence of this definition, the following statements are equivalent.

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x - r$ is a factor of f.
4. r is a solution to the equation $f(x) = 0$.

So the real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x) = 0$.

If r Is a Zero of Even Multiplicity

The sign of $f(x)$ does not change from one side to the other side of r.

The graph of f **touches** the x-axis at r.

If r Is a Zero of Odd Multiplicity

The sign of $f(x)$ changes from one side to the other side of r.

The graph of f **crosses** the x-axis at r.

Finding a Polynomial Function from Its Zeros

(a) Find a polynomial function of degree 3 whose zeros are -3, 2, and 5.

(b) Use a graphing utility to graph the polynomial found in part (a) to verify your result.

(a) If r is a real zero of a polynomial function f, then $x - r$ is a factor of f. This means that $x - (-3) = x + 3$, $x - 2$, and $x - 5$ are factors of f. As a result, any polynomial function of the form

$$f(x) = a(x + 3)(x - 2)(x - 5)$$

where a is a nonzero real number, qualifies. The value of a causes a stretch, compression, or reflection, but does not affect the x-intercepts of the graph. Do you know why?

(a) Find a polynomial function of degree 3 whose zeros are -3, 2, and 5.

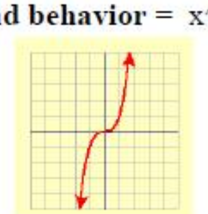
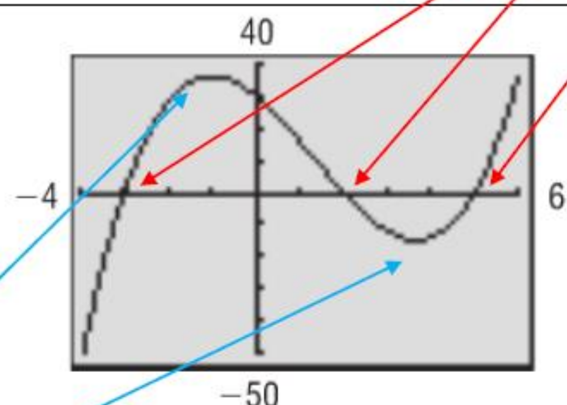
$$x - (-3) = x + 3,\ x - 2, \text{ and } x - 5$$

$$f(x) = a(x + 3)(x - 2)(x - 5)$$

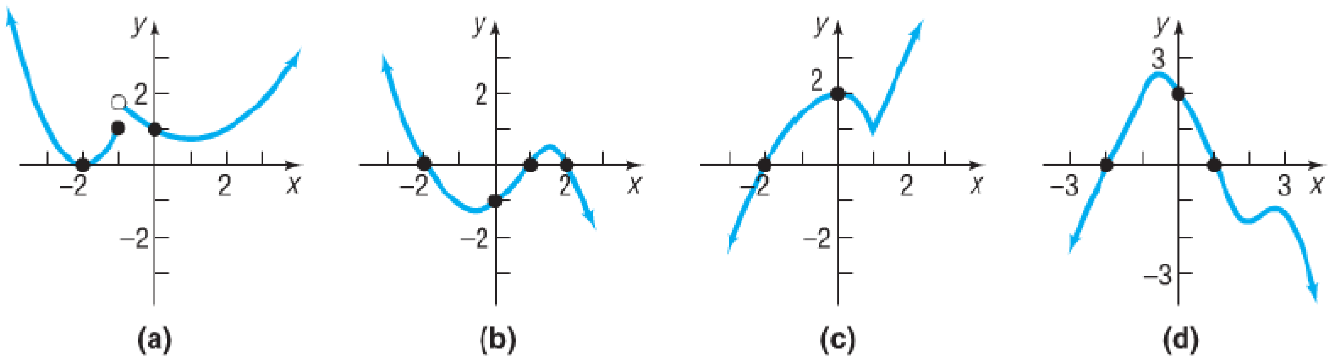
We choose to graph f with $a = 1$. Then

$$f(x) = (x + 3)(x - 2)(x - 5) = x^3 - 4x^2 - 11x + 30$$

Figure 10 shows the graph of f. Notice that the x-intercepts are -3, 2, and 5.

STEPS **1 – Determine End behavior** = x^3  **2—Determine the "y" intercept.** = (0,30) **Determine the "x" intercepts** = (-3,0),(2,0),(5,0) **3-- Determine multiplicity** "r" or x intercepts are odd so they cross the x axis. 4—**Determine turning points degree = n → n-1** n =3 3-1 = 2 turning points **5—Put it all together**	

Behavior near zero and turning points



(a) The graph in Figure 16(a) cannot be the graph of a polynomial function because of the gap that occurs at $x = -1$. Remember, the graph of a polynomial function is continuous—no gaps or holes.

(b) The graph in Figure 16(b) could be the graph of a polynomial function because the graph is smooth and continuous. It has three real zeros, at −2, at 1, and at 2. Since the graph has two turning points, the degree of the polynomial function must be at least 3.

(c) The graph in Figure 16(c) cannot be the graph of a polynomial function because of the cusp at $x = 1$. Remember, the graph of a polynomial function is smooth.

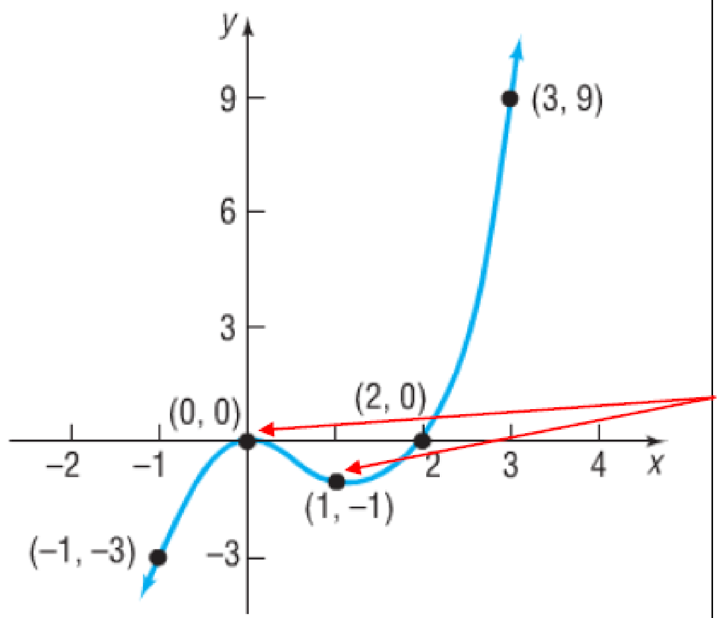
(d) The graph in Figure 16(d) could be the graph of a polynomial function. It has two real zeros, at −2 and at 1. Since the graph has three turning points, the degree of the polynomial function is at least 4.

Turning Points

If f is a polynomial function of degree n, then the graph of f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, the degree of f is at least n.

Figure 11



$f(x) = x^2(x - 2)$

Has two turning points thus the poly function is of degree 3 based on the turning point theorem.

n-1

3-1=2 turning points

Thus the degree of the functions is at least of degree 3.

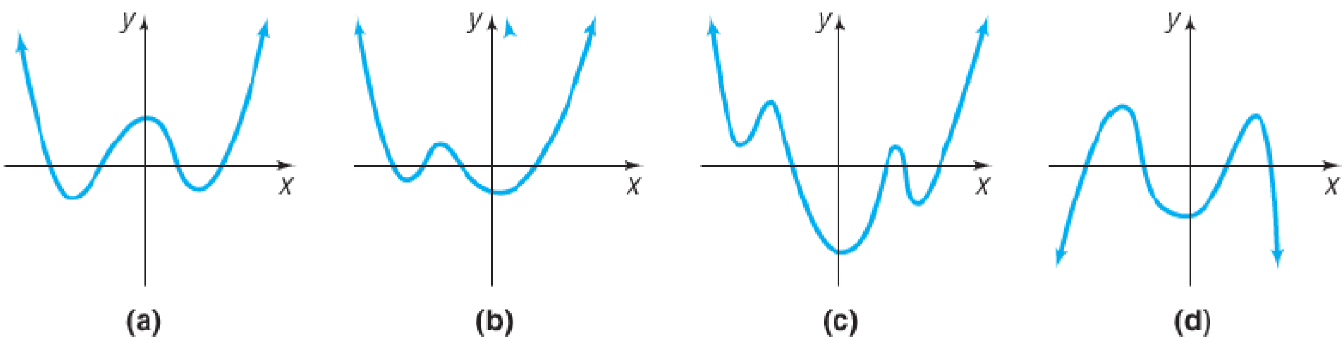
Turning points at (0,0) and
(1,-1) → exact (1.33, -1.18) between
(0,0)→(2,0) or [0,2]

changes from increasing to decreasing or vice versa at turning points.

Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 19 could be the graph of

$$f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6?$$



TYPE	A	B	C	D
END BEHAVIOUR	x^4	x^4	x^4	-x^4
MULTIPLICITY	4 odd	4 odd	4 odd	4 odd
Y INTERCEPTS	1 positive	1 negative	1 negative	1 negative
X INTERCEPTS	4	4	3	4
TURNING POINTS	3	3	5	3
DEGREE	4-1=3	4-1=3	6-1=5	4-3=1

The y-intercept of f is $f(0) = -6$. We can eliminate the graph in Figure 19(a), whose y-intercept is positive.

We don't have any methods for finding the x-intercepts of f, so we move on to investigate the turning points of each graph. Since f is of degree 4, the graph of f has at most 3 turning points. We eliminate the graph in Figure 19(c) since that graph has 5 turning points.

Now we look at end behavior. For large values of x, the graph of f will behave like the graph of $y = x^4$. This eliminates the graph in Figure 19(d), whose end behavior is like the graph of $y = -x^4$.

Only the graph in Figure 19(b) could be (and, in fact, is) the graph of $f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$.

End behavior

End Behavior

For large values of x, either positive or negative, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

resembles the graph of the power function

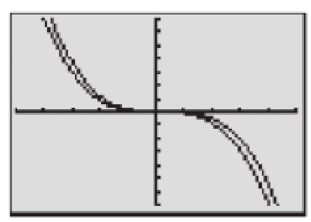
$$y = a_n x^n$$

For example, if $f(x) = -2x^3 + 5x^2 + x - 4$, then the graph of f will behave like the graph of $y = -2x^3$ for very large values of x, either positive or negative. We can see that the graphs of f and $y = -2x^3$ "behave" the same by considering Table 6 and Figure 17.

Table 6

x	$f(x)$	$y = -2x^3$
10	−1,494	−2,000
100	−1,949,904	−2,000,000
500	−248,749,504	−250,000,000
1,000	−1,994,999,004	−2,000,000,000

Figure 17



Analyzing the graph of a polynomial function

EXAMPLE 9 **How to Analyze the Graph of a Polynomial Function**

Analyze the graph of the polynomial function $f(x) = (2x + 1)(x - 3)^2$.

Step-by-Step Solution

Step 1: *Determine the end behavior of the graph of the function.*

Expand the polynomial to write it in the form

$$
\begin{aligned}
f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\
f(x) &= (2x + 1)(x - 3)^2 \\
&= (2x + 1)(x^2 - 6x + 9) \\
&= 2x^3 - 12x^2 + 18x + x^2 - 6x + 9 && \text{Multiply.} \\
&= 2x^3 - 11x^2 + 12x + 9 && \text{Combine like terms.}
\end{aligned}
$$

The polynomial function f is of degree 3. The graph of f behaves like $y = 2x^3$ for large values of $|x|$.

Step 2: *Find the x- and y-intercepts of the graph of the function.*

The y-intercept is $f(0) = 9$. To find the x-intercepts, we solve $f(x) = 0$.

$$
\begin{aligned}
f(x) &= 0 \\
(2x + 1)(x - 3)^2 &= 0 \\
2x + 1 = 0 \quad &\text{or} \quad (x - 3)^2 = 0 \\
x = -\frac{1}{2} \quad &\text{or} \quad x - 3 = 0 \\
&\qquad x = 3
\end{aligned}
$$

The x-intercepts are $-\frac{1}{2}$ and 3.

Step 3: *Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.*

The zeros of f are $-\frac{1}{2}$ and 3. The zero $-\frac{1}{2}$ is a zero of multiplicity 1, so the graph of f crosses the x-axis at $x = -\frac{1}{2}$. The zero 3 is a zero of multiplicity 2, so the graph of f touches the x-axis at $x = 3$.

Step 4: *Determine the maximum number of turning points on the graph of the function.*

Because the polynomial function is of degree 3 (Step 1), the graph of the function will have at most $3 - 1 = 2$ turning points.

Step 5: *Determine the behavior of the graph of f near each x-intercept.*

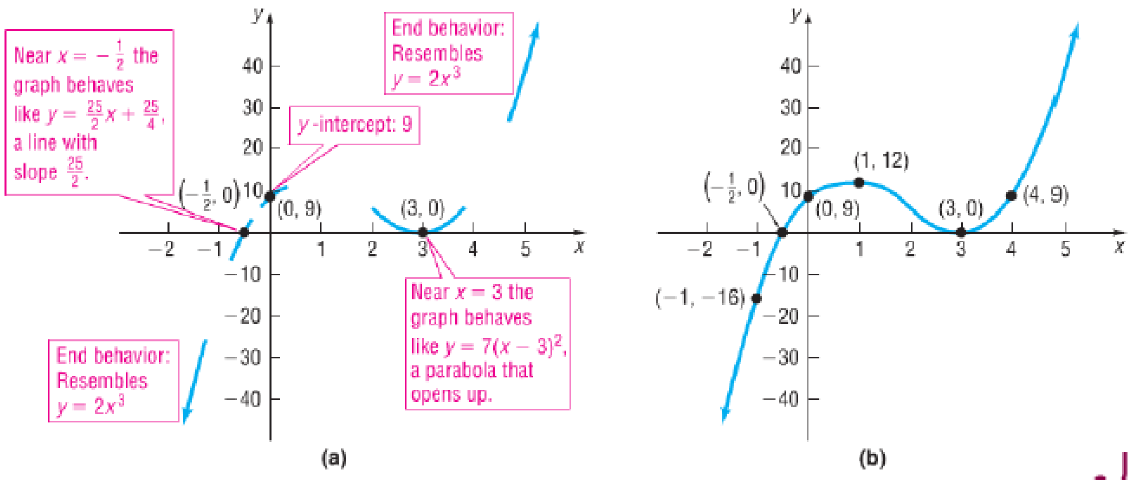
The two x-intercepts are $-\frac{1}{2}$ and 3.

$$
\begin{aligned}
\text{Near } -\tfrac{1}{2}: \quad f(x) &= (2x + 1)(x - 3)^2 \\
&\approx (2x + 1)\left(-\frac{1}{2} + 3\right)^2 \\
&= (2x + 1)\left(\frac{25}{4}\right) \\
&= \frac{25}{2}x + \frac{25}{4} && \text{A line with slope } \frac{25}{2} \\
\text{Near } 3: \quad f(x) &= (2x + 1)(x - 3)^2 \\
&\approx (2 \cdot 3 + 1)(x - 3)^2 \\
&= 7(x - 3)^2 && \text{A parabola that opens up}
\end{aligned}
$$

Step 6: Put all the information from Steps 1 through 5 together to obtain the graph of f.

Figure 20(a) illustrates the information obtained from Steps 1 through 5. We evaluate f at -1, 1, and 4 to help establish the scale on the y-axis. The graph of f is given in Figure 20(b).

Figure 20



SUMMARY Analyzing the Graph of a Polynomial Function

STEP 1: Determine the end behavior of the graph of the function.

STEP 2: Find the x- and y-intercepts of the graph of the function.

STEP 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

STEP 4: Determine the maximum number of turning points on the graph of the function.

STEP 5: Determine the behavior of the graph near each x-intercept.

STEP 6: Use the information in Steps 1 through 5 to draw a complete graph of the function.