CHAPTER 4 SULLIVAN 9th EDITION BOOK MATH 120 p2of2

Review Appendix A Sec A.3

Divide the polynomial $9x^5 - 3x^4 + 6x^3 + 7x^2 - 8x + 14$ by the polynomial $3x^2 + 2x - 1$.

Solution:

$$
\begin{array}{rrrrrrrrrrrrr}
 & & 3x^3 & - & 3x^2 & + & 5x & - & 2 & & & & \\
\cline{2-13}
3x^2 + 2x - 1 & \big) & 9x^5 & - & 3x^4 & + & 6x^3 & + & 7x^2 & - & 8x & + & 14 \\
 & -(& 9x^5 & + & 6x^4 & - & 3x^3\ ) & & & & & & \\
\cline{2-7}
 & & & - & 9x^4 & + & 9x^3 & + & 7x^2 & - & 8x & + & 14 \\
 & & & -(& -9x^4 & - & 6x^3 & + & 3x^2\ ) & & & & \\
\cline{4-9}
 & & & & & & 15x^3 & + & 4x^2 & - & 8x & + & 14 \\
 & & & & & -(& 15x^3 & + & 10x^2 & - & 5x\ ) & & \\
\cline{6-11}
 & & & & & & & - & 6x^2 & - & 3x & + & 14 \\
 & & & & & & & -(& -6x^2 & - & 4x & + & 2\ ) \\
\cline{8-13}
 & & & & & & & & & & x & + & 12
\end{array}
$$

DIVIDING POLYNOMIALS

Divide $4m^3 - 8m^2 + 4m + 6$ by $2m - 1$.

Solution

$4m^3$ divided by $2m$ is $2m^2$.

$-6m^2$ divided by $2m$ is $-3m$.

m divided by $2m$ is $\frac{1}{2}$.

$$
\begin{array}{r l}
 & 2m^2 - 3m + \frac{1}{2} \\
2m - 1 \,) & 4m^3 - 8m^2 + 4m + 6 \\
 & \underline{4m^3 - 2m^2} \quad \leftarrow 2m^2(2m - 1) = 4m^3 - 2m^2 \\
 & -6m^2 + 4m \quad \leftarrow \text{Subtract; bring down the next term.} \\
 & \underline{-6m^2 + 3m} \quad \leftarrow -3m(2m - 1) = -6m^2 + 3m \\
 & m + 6 \quad \leftarrow \text{Subtract; bring down the next term.} \\
 & \underline{m - \frac{1}{2}} \quad \leftarrow \frac{1}{2}(2m - 1) = m - \frac{1}{2} \\
 & \frac{13}{2} \quad \leftarrow \text{Subtract; the remainder is } \frac{13}{2}.
\end{array}
$$

To subtract, add the opposite.

Thus, $\dfrac{4m^3 - 8m^2 + 4m + 6}{2m - 1} = 2m^2 - 3m + \dfrac{1}{2} + \dfrac{\frac{13}{2}}{2m - 1}$.

Remember to add $\dfrac{\text{remainder}}{\text{divisor}}$

Find the quotient $(x^3 + 7x^2 + 7x - 11) \div (x + 5)$.

Solution

Write the division in the same format you use to divide whole numbers.

$$
\begin{array}{r l}
 & x^2 + 2x - 3 \\
x + 5 \,) & x^3 + 7x^2 + 7x - 11 \\
 & \underline{x^3 + 5x^2} \\
 & 2x^2 + 7x \\
 & \underline{2x^2 + 10x} \\
 & -3x - 11 \\
 & \underline{-3x - 15} \\
 & 4
\end{array}
$$

$x^3 \div x =$ x^2

Subtract x^2 $(x + 5) =$ $x^3 + 5x^2$.

Simplify and bring down $7x$.

Subtract $2x$ $(x + 5) =$ $2x^2 + 10x$.

Simplify and bring down -11.

Subtract -3 $(x + 5) =$ $-3x - 15$.

Remainder

The result is written as $x^2 + 2x - 3 + \dfrac{4}{x + 5}$.

EXAMPLE 10 DIVIDING POLYNOMIALS WITH MISSING TERMS

Divide $3x^3 - 2x^2 - 150$ by $x^2 - 4$.

Solution Both polynomials have missing first-degree terms. Insert each missing term with a 0 coefficient.

$$
\begin{array}{r|l}
 & 3x - 2 \\ \hline
x^2 + 0x - 4 & 3x^3 - 2x^2 + 0x - 150 \\
 & \underline{3x^3 + 0x^2 - 12x} \\
 & -2x^2 + 12x - 150 \\
 & \underline{-2x^2 + 0x + 8} \\
 & 12x - 158
\end{array}
$$

Missing term

Missing term

Insert placeholders for missing terms.

Remainder

The division process ends when the remainder is 0 or the degree of the remainder is less than that of the divisor. Since $12x - 158$ has lesser degree than the divisor $x^2 - 4$, it is the remainder. Thus,

$$\frac{3x^3 - 2x^2 - 150}{x^2 - 4} = 3x - 2 + \frac{12x - 158}{x^2 - 4}.$$

$-3x^3 + 7x^2 + 8x + 1$ by $x - 3$

$$
\begin{array}{r|l}
 & \quad -3x^2 \quad - \quad 2x \quad + \quad 2 \\
\hline
x - 3 & -3x^3 + 7x^2 + 8x + 1 \\
 & -(-3x^3 + 9x^2) \\
\cline{2-2}
 & \qquad -2x^2 + 8x + 1 \\
 & \qquad -(-2x^2 + 6x) \\
\cline{2-2}
 & \qquad\qquad 2x + 1 \\
 & \qquad\qquad -(2x - 6) \\
\cline{2-2}
 & \qquad\qquad\qquad 7
\end{array}
$$

Synthetic Division requires the divisor to be in the format of "(x-k)"

Take the opposite of (x – 3)←→ (x - (+3))

3	–3	7	8	1
		–9	–6	6
	–3	–2	2	7

Take the opposite of (x – 3) ←→ (x - (+3))

3	–3	7	8	1
		$7 + (-9)$	$8 + (-6)$	$1 + 6$
	$3(-3)$	–9	–6	6
		$3(-2)$	$3(2)$	
	–3	–2	2	7

Example. Use synthetic division to perform the indicated operation: $\dfrac{x^4 - 3x^3 - 5x^2 + 2x - 18}{x + 2}$

Solution. The divisor, x + 2, must be written as a difference, x – (-2), to determine that the divider is –2.

-2	1	-3	-5	2	-18
		-2	10	-10	16
	1	-5	5	-8	-2

$$\frac{x^4 - 3x^3 - 5x^2 + 2x - 16}{x + 2} = x^3 - 5x^2 + 5x - 8 + \frac{-2}{x + 2}$$

f(x) = d(x) · q(x) + r(x)/d(x)

Dividend = Divisor · Quotient + Remainder/Divisor

4.2 Rational Functions, [Polynomial division (Review Appendix A Sec A.3)]

Ratios of polynomials are called rational functions. They include:

$$R(x) = \frac{x^2 - 4}{x^2 + x + 1} \qquad F(x) = \frac{x^3}{x^2 - 4} \qquad G(x) = \frac{3x^2}{x^4 - 1}$$

A **rational function** is a function of the form

$$\boxed{R(x) = \frac{p(x)}{q(x)}}$$

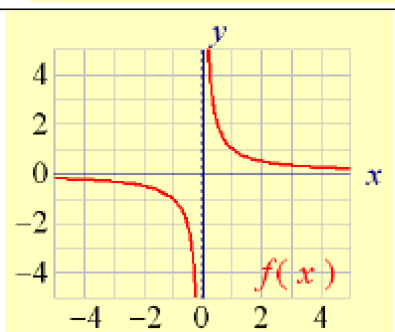
where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Rational Functions

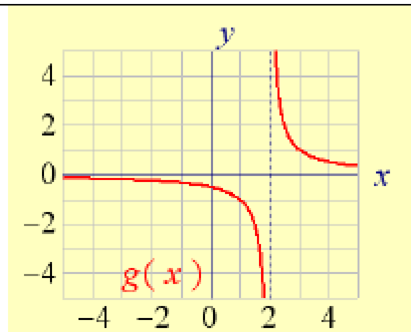
A **rational function** is a function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

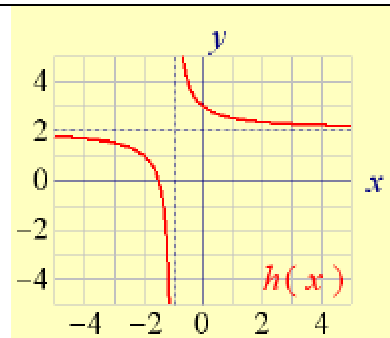
where $p(x)$ and $q(x)$ are both polynomial functions and $q(x) \neq 0$. Of course, even though q is not allowed to be identically zero, there will often be values of x for which $q(x)$ is zero, and at these values the fraction is undefined. Consequently, the **domain of** f is the set $\{\ x \mid q(x) \neq 0\ \}$.



$f(x) = \frac{1}{x}$



$g(x) = \frac{1}{x-2}$



$h(x) = \frac{1}{x+1} + 2$

$\frac{1}{x+1} + \frac{2x+2}{x+1} = \frac{2x+3}{x+1}$

Graphing y = 1 / x^2 and transformations

Graphing $y = \frac{1}{x^2}$

Graphing $y = \frac{1}{x^2}$

Analyze the graph of $H(x) = \frac{1}{x^2}$.

The domain of $H(x) = \frac{1}{x^2}$ is the set of all real numbers x except 0. The graph has no y-intercept, because x can never equal 0. The graph has no x-intercept because the equation $H(x) = 0$ has no solution. Therefore, the graph of H will not cross or touch either of the coordinate axes. Because

$$H(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = H(x)$$

H is an even function, so its graph is symmetric with respect to the y-axis.

Table 9 shows the behavior of $H(x) = \frac{1}{x^2}$ for selected positive numbers x. (We will use symmetry to obtain the graph of H when $x < 0$.) From the first three rows of Table 9, we see that, as the values of x approach (get closer to) 0, the values of $H(x)$ become larger and larger positive numbers, so H is unbounded in the positive direction. We use limit notation, $\lim_{x \to 0} H(x) = \infty$, read "the limit of $H(x)$ as x approaches zero equals infinity," to mean that $H(x) \to \infty$ as $x \to 0$.

Look at the last four rows of Table 9. As $x \to \infty$, the values of $H(x)$ approach 0 (the end behavior of the graph). In calculus, this is symbolized by writing $\lim_{x \to \infty} H(x) = 0$. Figure 28 shows the graph. Notice the use of red dashed lines to convey the ideas discussed above.

Figure 28

$H(x) = \frac{1}{x^2}$

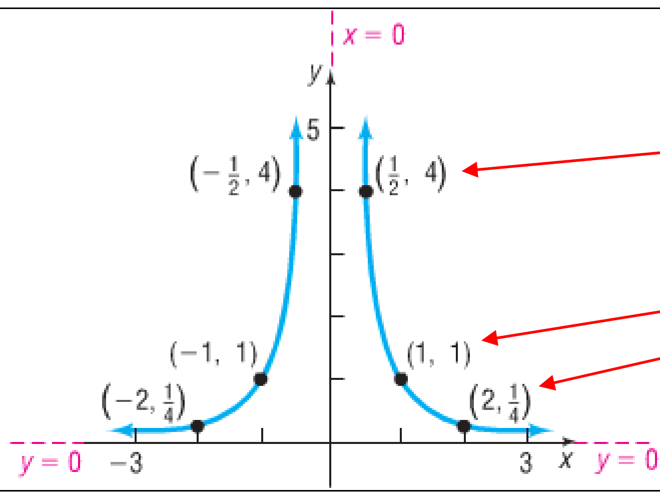


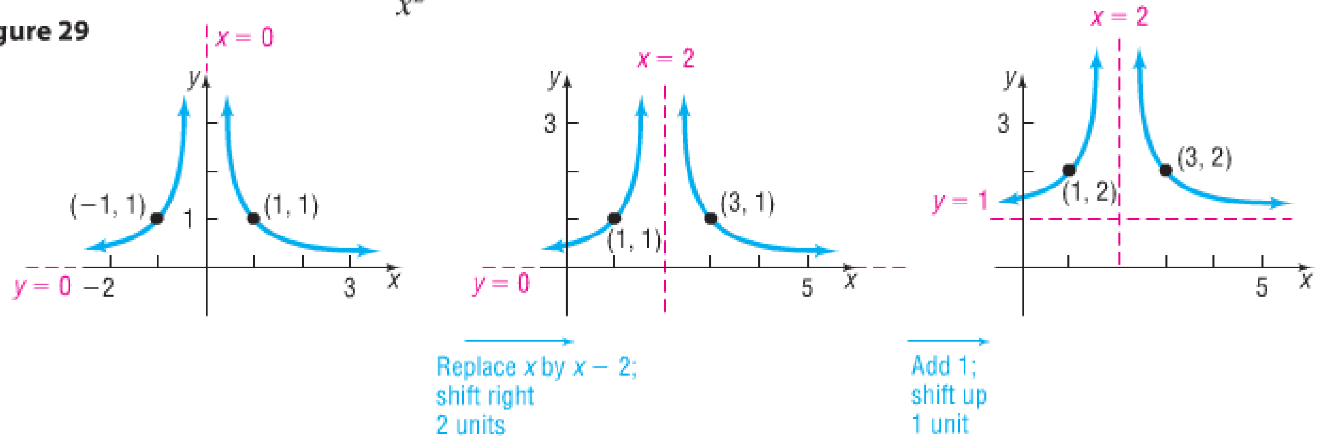
Table 9

x	$H(x) = \frac{1}{x^2}$
$\frac{1}{2}$	4
$\frac{1}{100}$	10,000
$\frac{1}{10{,}000}$	100,000,000
1	1
2	$\frac{1}{4}$
100	$\frac{1}{10{,}000}$
10,000	$\frac{1}{100{,}000{,}000}$

Graph the rational function: $R(x) = \dfrac{1}{(x-2)^2} + 1$

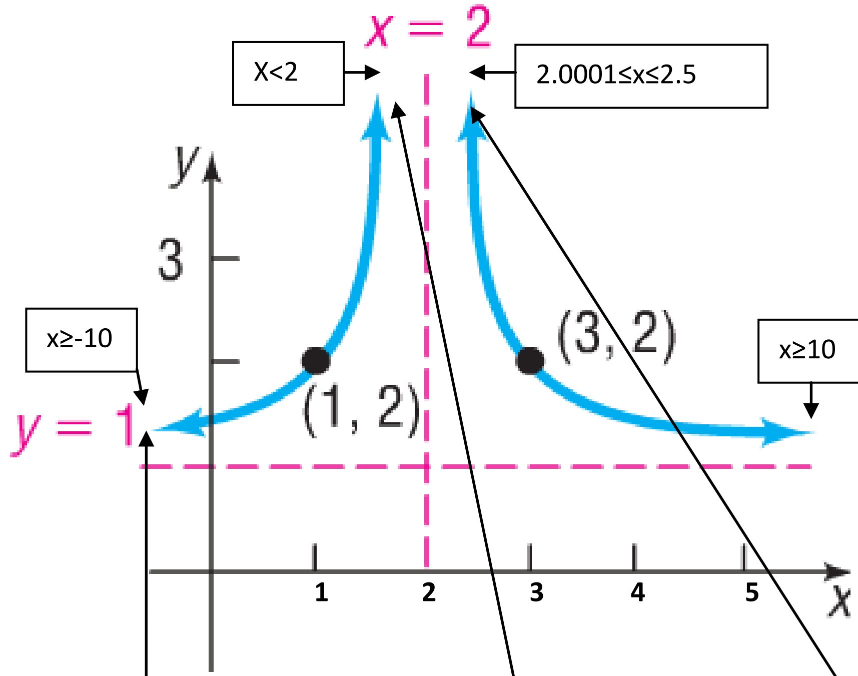
Solution The domain of R is the set of all real numbers except $x = 2$. To graph R, start with the graph of $y = \dfrac{1}{x^2}$. See Figure 29 for the steps.

Figure 29



(a) $y = \dfrac{1}{x^2}$ **(b)** $y = \dfrac{1}{(x-2)^2}$ **(c)** $y = \dfrac{1}{(x-2)^2} + 1$

Asymptotes – a line approached by a curve in the limit as the curve approaches infinity.



$$R(x) = y = \frac{1}{(x-2)^2} + 1$$

Vertical line x =2 and horizontal line y =1 in the above figure.

HORIZONTAL ASYMPTOTE

x	*R*(*x*)
10	1.0156
100	1.0001
1000	1.000001
10,000	1.00000001

(a)

The higher the value of x = 10, 100, 1000, 10000 the R(x) or y = keeps reaching the value of 1 in the positive direction.

Limits are "x" to positive infinity

Horizontal = x→+infinity R(x =1)

HORIZONTAL ASYMPTOTE

x	*R*(*x*)
−10	1.0069
−100	1.0001
−1000	1.000001
−10,000	1.00000001

(b)

The higher the value of x = -10, -100, -1000, -10000 the

R(x) or y = keeps reaching the value of 1 in the negative direction.

Limits are "x" to negative infinity

R(x)=1 or y = 1

Horizontal = x→-infinity

VERTICAL ASYMPTOTE

x	*R*(*x*)
1.5	5
1.9	101
1.99	10,001
1.999	1,000,001
1.9999	100,000,001

(c)

x=2 is not part of the domain but it is important to see the graphs behavior as it approaches x =2

As "x"< 2

the value for "y" or R(x) increases to infinity.

x =1.99 results y= 100,000,001

$x \to 2^{--}$

Vertical = "y" or R(x) →+inf

VERTICAL ASYMPTOTE

x	*R*(*x*)
2.5	5
2.1	101
2.01	10,001
2.001	1,000,001
2.0001	100,000,001

(d)

As "x"> 2

the value for "y" or R(x) increases to infinity.

x = results y= 100,000,001

To see the effect better

2.0001 ≤ "x" ≤ 2.5

the value for $x \to 2^{+}$

Vertical = "y" or R(x) →+inf

Vertical & horizontal (oblique) asymptotes of a rational function

Vertical Asymptotes

The vertical line $x = c$ is a **vertical asymptote** of a function f if $f(x)$ increases in magnitude without bound as x approaches c. Examples of vertical asymptotes appear in Figure 2. The graph of a rational function cannot intersect a vertical asymptote.

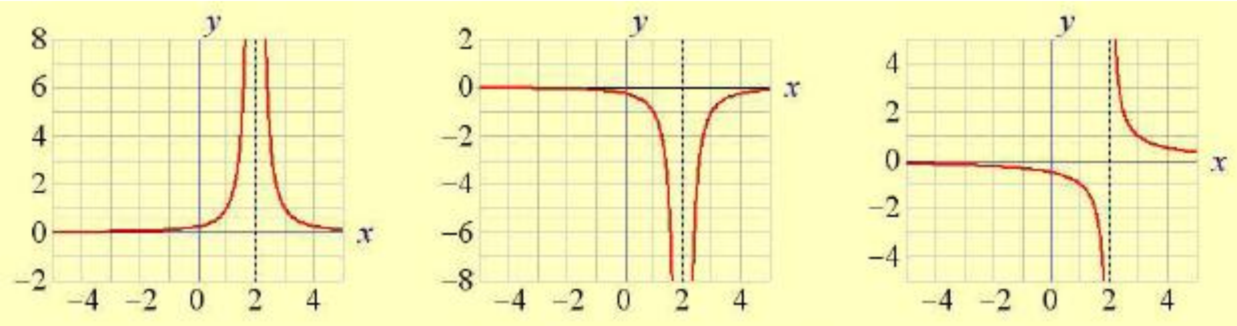


Figure 2: Vertical Asymptotes

Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \dfrac{x + 3}{x - 1}$ (b) $R(x) = \dfrac{x}{x^2 - 4}$

(c) $H(x) = \dfrac{x^2}{x^2 + 1}$ (d) $G(x) = \dfrac{x^2 - 9}{x^2 + 4x - 21}$

(a) F is in lowest terms and the only zero of the denominator is 1. The line $x = 1$ is the vertical asymptote of the graph of F.

(b) R is in lowest terms and the zeros of the denominator $x^2 - 4$ are -2 and 2. The lines $x = -2$ and $x = 2$ are the vertical asymptotes of the graph of R.

(c) H is in lowest terms and the denominator has no real zeros, because the equation $x^2 + 1 = 0$ has no real solutions. The graph of H has no vertical asymptotes.

(d) Factor the numerator and denominator of $G(x)$ to determine if it is in lowest terms.

$$G(x) = \frac{x^2 - 9}{x^2 + 4x - 21} = \frac{(x + 3)(x - 3)}{(x + 7)(x - 3)} = \frac{x + 3}{x + 7} \qquad x \neq 3$$

The only zero of the denominator of $G(x)$ in lowest terms is -7. The line $\boxed{x = -7}$ is the only vertical asymptote of the graph of G.

a. $f(x) = \dfrac{x^2 - 6}{x^2 + 1}$ **b.** $g(x) = \dfrac{x^2 - 1}{x^2 + 2x - 3}$

Solutions:

a. In this case, there are no common terms in the numerator and denominator. Solving for x in the denominator in an attempt to find the asymptotes, we end up with $x = \sqrt{-1}$. Since we cannot have imaginary numbers in the Cartesian Coordinate System, f has no vertical asymptotes.

b. As usual, in order to determine the domain of the rational function we have to factor the denominator. And after making note of the domain, we will look for common factors to cancel, so we may as well factor the numerator, if possible:

i. $g(x) = \dfrac{x^2 - 1}{x^2 + 2x - 3} = \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x+3)} = \dfrac{x+1}{x+3}$

ii. What we see in this example is that the domain is $\{\ x \mid x \neq -3 \text{ and } x \neq 1\ \}$. Thus, the equation of the vertical asymptote is $x = -3$.

Horizontal Asymptotes

The horizontal line $y = c$ is a **horizontal asymptote** of a function f if $f(x)$ approaches the value c as $x \to -\infty$ or as $x \to \infty$. Examples of horizontal asymptotes appear in Figure 3. The graph of a rational function may intersect a horizontal asymptote, often near the origin.

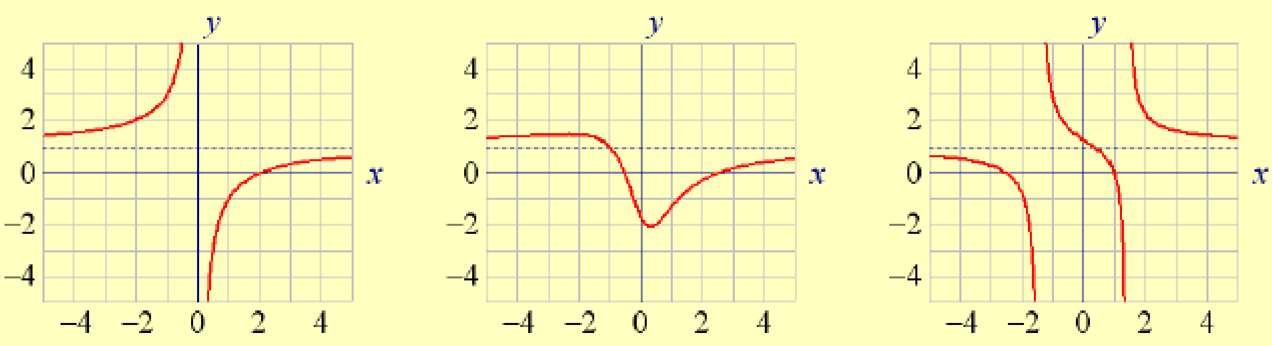


Figure 3: Horizontal Asymptotes

Finding a Horizontal Asymptote

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

Since the degree of the numerator, 1, is less than the degree of the denominator, 2, the rational function R is proper. The line $y = 0$ is a horizontal asymptote of the graph of R.

$$f(x) = \frac{6x + 1}{5x^2 - 2x + 3}$$

The degree in the numerator is less than the degree in the denominator thus the rational function is proper and the line y = 0 is a horizontal asymptote of the graph R

In symbols, $f(x) \to 0$ as $x \to \pm\infty$.

Oblique Asymptotes

A non-vertical, non-horizontal line may also be an asymptote of a function f. Examples of **oblique** (or **slant**) asymptotes appear in Figure 4. Again, the graph of a rational function may intersect an oblique asymptote.

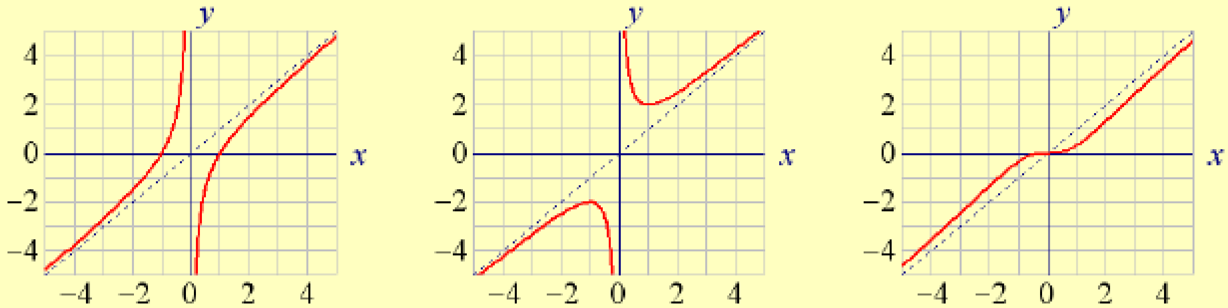
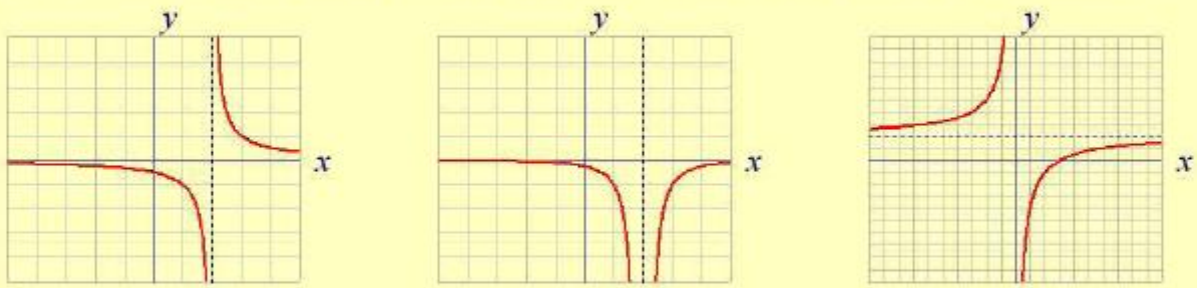


Figure 4: Oblique Asymptotes

Asymptote Notation

The notation $x \to c^-$ is used in describing the behavior of a graph as x approaches the value c from the left (the negative side). The notation $x \to c^+$ is used in describing behavior as x approaches c from the right (the positive side). The notation $x \to c$ is used in describing behavior that is the same on both sides of c.

Figure 5 illustrates how the above notation can be used to describe the behavior of functions.



$f(x) \to -\infty$ as $x \to 2^-$
$f(x) \to \infty$ as $x \to 2^+$

$g(x) \to -\infty$ as $x \to 2$

$h(x) \to 2$ as $x \to -\infty$
$h(x) \to 2$ as $x \to \infty$

Figure 5: Asymptote Notation

Finding a Horizontal Asymptote

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

n Since the degree of the numerator, 1, is less than the degree of the denominator, 2, the rational function R is proper. The line $y = 0$ is a horizontal asymptote of the graph of R.

To see why $y = 0$ is a horizontal asymptote of the function R in Example 5, we investigate the behavior of R as $x \to -\infty$ and $x \to \infty$. When $|x|$ is very large, the numerator of R, which is $x - 12$, can be approximated by the power function $y = x$, while the denominator of R, which is $4x^2 + x + 1$, can be approximated by the power function $y = 4x^2$. Applying these ideas to $R(x)$, we find

$$R(x) = \frac{x - 12}{4x^2 + x + 1} \underset{\substack{\uparrow \\ \text{For } |x| \text{ very large}}}{\approx} \frac{x}{4x^2} = \frac{1}{4x} \underset{\substack{\uparrow \\ \text{As } x \to -\infty \text{ or } x \to \infty}}{\to} 0$$

This shows that the line $y = 0$ is a horizontal asymptote of the graph of R.

If a rational function $R(x) = \frac{p(x)}{q(x)}$ is **improper,** that is, if the degree of the numerator is greater than or equal to the degree of the denominator, we use long division to write the rational function as the sum of a polynomial $f(x)$ (the quotient) plus a proper rational function $\frac{r(x)}{q(x)}$ ($r(x)$ is the remainder). That is, we write

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

where $f(x)$ is a polynomial and $\frac{r(x)}{q(x)}$ is a proper rational function. Since $\frac{r(x)}{q(x)}$ is proper, $\frac{r(x)}{q(x)} \to 0$ as $x \to -\infty$ or as $x \to \infty$. As a result,

$$R(x) = \frac{p(x)}{q(x)} \to f(x) \qquad \text{as } x \to -\infty \text{ or as } x \to \infty$$

The possibilities are listed next.

1. If $f(x) = b$, a constant, the line $y = b$ is a horizontal asymptote of the graph of R.

Slanted 45 degrees

2. If $f(x) = ax + b$, $a \neq 0$, the line $y = ax + b$ is an oblique asymptote of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

We illustrate each of the possibilities in Examples 6, 7, and 8.

EX 6

Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Since the degree of the numerator, 4, is greater than the degree of the denominator, 3, the rational function H is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r|l}
 & 3x + 3 \\
\hline
x^3 - x^2 + 1 & 3x^4 \qquad\qquad - x^2 \\
 & \underline{3x^4 - 3x^3 \qquad\quad + 3x} \\
 & \qquad 3x^3 - x^2 - 3x \\
 & \qquad \underline{3x^3 - 3x^2 \qquad + 3} \\
 & \qquad\qquad 2x^2 - 3x - 3
\end{array}$$

As a result,

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1} = 3x + 3 + \frac{2x^2 - 3x - 3}{x^3 - x^2 + 1}$$

As $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 - 3x - 3}{x^3 - x^2 + 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $H(x) \to 3x + 3$. We conclude that the graph of the rational function H has an oblique asymptote $y = 3x + 3$.

$$f(x) = \frac{7x^3 + 2x - 1}{x^2 + 4x}$$

Since the degree in the numerator is greater than the numerator then do long division.

EX 7

Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

Since the degree of the numerator, 2, equals the degree of the denominator, 2, the rational function R is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r} 2 \\ 4x^2 - 1 \overline{\smash{)}\, 8x^2 - x + 2} \\ \underline{8x^2 - 2} \\ - x + 4 \end{array}$$

As a result,

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1} = 2 + \frac{-x + 4}{4x^2 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{-x + 4}{4x^2 - 1} \approx \frac{-x}{4x^2} = \frac{-1}{4x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $R(x) \to 2$. We conclude that $y = 2$ is a horizontal asymptote of the graph.

In Example 7, notice that the quotient 2 obtained by long division is the quotient of the leading coefficients of the numerator polynomial and the denominator polynomial $\left(\frac{8}{4}\right)$. This means that we can avoid the long division process for rational functions where the numerator and denominator *are of the same degree* and conclude that the quotient of the leading coefficients will give us the horizontal asymptote.

$$f(x) = \frac{6x^2 - 3x + 2}{3x^2 + 5x - 17}$$

In symbols, $f(x) \to 2$ as $x \to \pm\infty$.

If the degree are equal then divide the coefficient to obtain the horizontal asymptote.

EX 8

Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

n

Since the degree of the numerator, 5, is greater than the degree of the denominator, 3, the rational function G is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r} 2x^2 - 1 \\ x^3 - 1 \overline{)\, 2x^5 - x^3 + 2} \\ \underline{2x^5 - 2x^2 } \\ -x^3 + 2x^2 + 2 \\ \underline{-x^3 + 1} \\ 2x^2 + 1 \end{array}$$

As a result,

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1} = 2x^2 - 1 + \frac{2x^2 + 1}{x^3 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 + 1}{x^3 - 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $G(x) \to 2x^2 - 1$. We conclude that, for large values of $|x|$, the graph of G approaches the graph of $y = 2x^2 - 1$. That is, the graph of G will look like the graph of $y = 2x^2 - 1$ as $x \to -\infty$ or $x \to \infty$. Since $y = 2x^2 - 1$ is not a linear function, G has no horizontal or oblique asymptote.

$$f(x) = \frac{3x^5 - 2x^3 + 7x^2 - 1}{4x^3 + 19x^2 - 3x + 5}$$

If the degree in the numerator is greater than the degree in the denominator and it is greater than 1 then the function is not oblique or horizontal.

SUMMARY Finding a Horizontal or Oblique Asymptote of a Rational Function

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).
2. If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then R is improper. Here long division is used.
   (a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
   (b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.
   (c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for very large values of $|x|$, the graph of R will behave like the graph of the quotient.

Note: The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote.

Equations for Horizontal and Oblique Asymptotes

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function, where p is an n^{th} degree polynomial with leading coefficient a_n and q is an m^{th} degree polynomial with leading coefficient b_m.

Then:

1. If $n < m$, the horizontal line $y = 0$ (the x-axis) is the horizontal asymptote for f.
2. If $n = m$, the horizontal line $y = \frac{a_n}{b_m}$ is the horizontal asymptote for f.
3. If $n = m + 1$, the line $y = g(x)$ is an oblique asymptote for f, where g is the quotient polynomial obtained by dividing p by q (the remainder polynomial is irrelevant).
4. If $n > m + 1$, there is no straight-line horizontal or oblique asymptote for f.

Find the equation for the horizontal or oblique asymptote of the following functions.

a. $g(x) = \dfrac{x+1}{x^2+2x-15}$ **b.** $h(x) = \dfrac{x^3-27}{2x+2}$

Solutions:

a. Because the numerator has a degree less than the degree of the denominator of g, the asymptote is the horizontal line $y = 0$.

b. Because the degree of the numerator is greater than one plus the degree of the denominator, we know h has no oblique or horizontal asymptote.

a. $f(x) = \dfrac{x^3-x^2+4}{x+5}$ **Vertical** $x =$ ☐ **horizontal/oblique** $y =$ ☐

b. $g(x) = \dfrac{2x-3}{x^2-9}$ **Vertical** $x =$ ☐ **horizontal/oblique** $y =$ ☐

c. $h(x) = \dfrac{x^2+x-7}{2x^2-8}$ **Vertical** $x =$ ☐ **horizontal/oblique** $y =$ ☐

a. $f(x) = \dfrac{x^3-x^2+4}{x+5}$ **Vertical** $x = \boxed{-5}$ **horizontal/oblique** $y = \boxed{\text{None}}$

If the degree in the numerator is greater than the degree in the denominator and it is greater than 1 then the function is not oblique or horizontal.

b. $g(x) = \dfrac{2x-3}{x^2-9}$ **Vertical** $x = \boxed{3,-3}$ **horizontal/oblique** $y = \boxed{0}$

"x" Vertical asymptote to make the denominator equal to zero.

The rational equation is proper because the degree for the numerator is less than the degree in the denominator.

c. $h(x) = \dfrac{x^2+x-7}{2x^2-8}$ **Vertical** $x = \boxed{2,-2}$ **horizontal/oblique** $y = \dfrac{\boxed{1}}{\boxed{2}}$

The rational equation is improper because the degree for the numerator is equal to the degree in the denominator.

Same degrees divide coefficients for "y" horizontal asymptote

"x" Vertical asymptote to make the denominator equal to zero.

4.3 Properties of rational functions (very brief)

EXAMPLE 5 **Analyzing the Graph of a Rational Function with a Hole**

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

Solution **STEP 1:** Factor R and obtain

$$R(x) = \frac{(2x-1)(x-2)}{(x+2)(x-2)}$$

The domain of R is $\{x | x \neq -2, x \neq 2\}$.

STEP 2: In lowest terms,

$$R(x) = \frac{2x-1}{x+2} \qquad x \neq -2, x \neq 2$$

STEP 3: The y-intercept is $R(0) = -\frac{1}{2}$. Plot the point $\left(0, -\frac{1}{2}\right)$.
The graph has one x-intercept: $\frac{1}{2}$.

Near $\frac{1}{2}$: $R(x) = \dfrac{2x-1}{x+2} \approx \dfrac{2x-1}{\frac{1}{2}+2} = \dfrac{2}{5}(2x-1)$

Plot the point $\left(\frac{1}{2}, 0\right)$ showing a line with positive slope.

STEP 4: Since $x + 2$ is the only factor of the denominator of $R(x)$ *in lowest terms*, the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$. Graph the line $x = -2$ using dashes.

STEP 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. Graph the line $y = 2$ using dashes.

To find out whether the graph of R intersects the horizontal asymptote $y = 2$, we solve the equation $R(x) = 2$.

$$\begin{aligned} R(x) = \frac{2x-1}{x+2} &= 2 \\ 2x - 1 &= 2(x+2) \\ 2x - 1 &= 2x + 4 \\ -1 &= 4 \qquad \text{Impossible} \end{aligned}$$

The graph does not intersect the line $y = 2$.

STEP 6: Look at the factored expression for R in Step 1. The real zeros of the numerator and denominator, $-2, \frac{1}{2}$, and 2, divide the x-axis into four intervals:

$$(-\infty, -2) \qquad \left(-2, \frac{1}{2}\right) \qquad \left(\frac{1}{2}, 2\right) \qquad (2, \infty)$$

Construct Table 15. Plot the points in Table 15.

5

	-2	$1/2$	2	x
Interval	$(-\infty, -2)$	$\left(-2, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 2\right)$	$(2, \infty)$
Number chosen	-3	-1	1	3
Value of *R*	$R(-3) = 7$	$R(-1) = -3$	$R(1) = \frac{1}{3}$	$R(3) = 1$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Above x-axis
Point on graph	$(-3, 7)$	$(-1, -3)$	$\left(1, \frac{1}{3}\right)$	$(3, 1)$

COMMENT The coordinates of the hole were obtained by evaluating R in lowest terms at 2. R in lowest terms is $\frac{2x-1}{x+2}$, which, at $x = 2$, is $\frac{2(2)-1}{2+2} = \frac{3}{4}$. ■

Step 7: • From Table 15 we know that the graph of R is above the x-axis for $x < -2$.

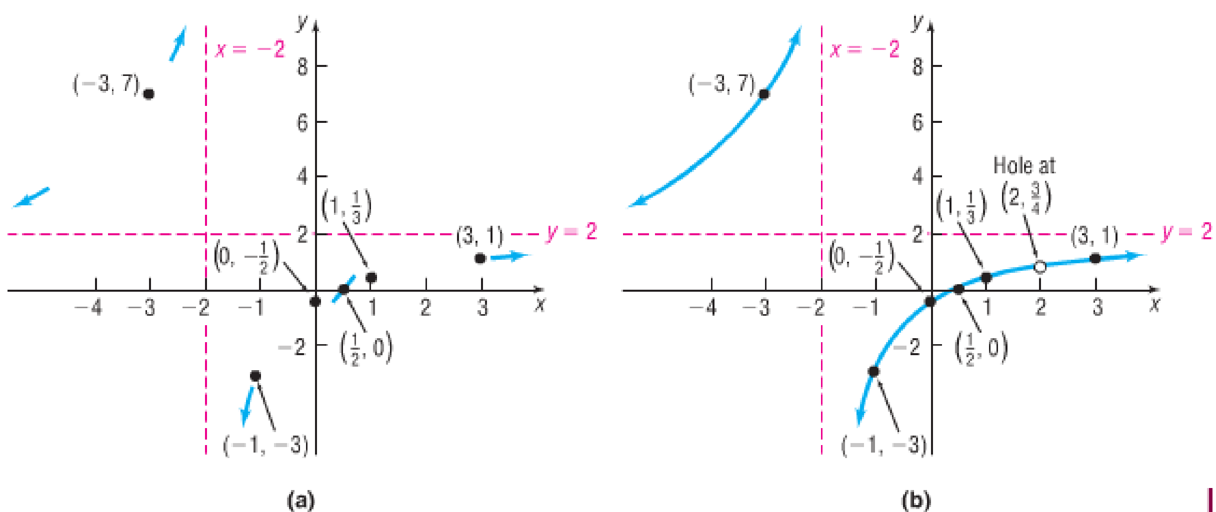
From Step 5 we know that the graph of R does not intersect the asymptote $y = 2$. Therefore, the graph of R will approach $y = 2$ from above as $x \to -\infty$ and will approach the vertical asymptote $x = -2$ at the top from the left.

Since the graph of R is below the x-axis for $-2 < x < \frac{1}{2}$, the graph of R will approach $x = -2$ at the bottom from the right.

Finally, since the graph of R is above the x-axis for $x > \frac{1}{2}$ and does not intersect the horizontal asymptote $y = 2$, the graph of R will approach $y = 2$ from below as $x \to \infty$. See Figure 40(a).

Step 8: See Figure 40(b) for the complete graph. Since R is not defined at 2, there is a hole at the point $\left(2, \frac{3}{4}\right)$.

Figure 40



Exploration

Graph $R(x) = \frac{2x^2 - 5x + 2}{x^2 - 4}$. Do you see the hole at $\left(2, \frac{3}{4}\right)$? TRACE along the graph. Did you obtain an ERROR at $x = 2$? Are you convinced that an algebraic analysis of a rational function is required in order to accurately interpret the graph obtained with a graphing utility?

As Example 5 shows, **the zeros of the denominator of a rational function give rise to either vertical asymptotes or holes in the graph.**

4.4 Rational Inequalities (very brief)

Solving a rational inequality using a graph. // olving a rational inequality algebraically.

4.5 The real zeros of polynomial functions

Rational zeros theorem (RZT)

Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function

The next result, called the **Rational Zeros Theorem,** provides information about the rational zeros of a polynomial *with integer coefficients*.

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0, \quad a_0 \neq 0}$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

SUMMARY Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of real zeros.

STEP 2: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 2 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

Special products	Factoring by groups
SPECIAL PRODUCT PATTERNS **Sum and Difference** / **Example** $(a + b)(a - b) = a^2 - b^2$ $(x + 6)(x - 6) = \underline{x^2 - 36}$ **Square of a Binomial** / **Example** $(a + b)^2 = a^2 + 2ab + b^2$ $(x + 5)^2 = \underline{x^2 + 10x + 25}$ $(a - b)^2 = a^2 - 2ab + b^2$ $(2x - 3)^2 = \underline{4x^2 - 12x + 9}$ **Cube of a Binomial** / **Example** $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ $(x + 3)^3 = \underline{x^3} + \underline{9}\,x^2 + \underline{27}\,x + \underline{27}$ $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ $(x - 4)^3 = \underline{x^3} - \underline{12}\,x^2 + \underline{48}\,x - \underline{64}$	**Factor the polynomial $x^3 - 3x^2 - 36x + 108$.** **Solution** $x^3 - 3x^2 - 36x + 108$ $= (\underline{x^3 - 3x^2}) + (\underline{-36x + 108})$ **Group terms.** $= x^2(\underline{x - 3}) + (-36)(\underline{x - 3})$ **Factor each group.** $= (x^2 - 36)(\underline{x - 3})$ **Distributive property** $= (\underline{x + 6})(\underline{x - 6})(\underline{x - 3})$ **Difference of two squares**

EXAMPLE 4 **How to Find the Real Zeros of a Polynomial Function**

Find the real zeros of the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$. Write f in factored form.

Step-by-Step Solution

Step 1: Use the degree of the polynomial to determine the maximum number of zeros.

Since f is a polynomial of degree 3, there are at most three real zeros.

Step 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros. Use the Factor Theorem to determine if each potential rational zero is a zero. If it is, use synthetic division or long division to factor the polynomial function. Repeat Step 2 until all the zeros of the polynomial function have been identified and the polynomial function is completely factored.

List the potential rational zeros obtained in Example 3:

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

From our list of potential rational zeros, we will test 6 to determine if it is a zero of f. Because $f(6) = 780 \neq 0$, we know that 6 is not a zero of f. Now, let's test if -6 is a zero. Because $f(-6) = 0$, we know that -6 is a zero and $x - (-6) = x + 6$ is a factor of f. Use long division or synthetic division to factor f. (We will not show the division here, but you are encouraged to verify the results shown.) After dividing f by $x + 6$, the quotient is $2x^2 - x - 1$, so

$$\begin{aligned} f(x) &= 2x^3 + 11x^2 - 7x - 6 \\ &= (x + 6)(2x^2 - x - 1) \end{aligned}$$

Now any solution of the equation $2x^2 - x - 1 = 0$ will be a zero of f. We call the equation $2x^2 - x - 1 = 0$ a **depressed equation** of f. Because any solution to the equation $2x^2 - x - 1 = 0$ is a zero of f, we work with the depressed equation to find the remaining zeros of f.

The depressed equation $2x^2 - x - 1 = 0$ is a quadratic equation with discriminant $b^2 - 4ac = (-1)^2 - 4(2)(-1) = 9 > 0$. The equation has two real solutions, which can be found by factoring.

$$\begin{aligned} 2x^2 - x - 1 = (2x + 1)(x - 1) &= 0 \\ 2x + 1 = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = -\frac{1}{2} \quad &\text{or} \quad x = 1 \end{aligned}$$

The zeros of f are $-6, -\frac{1}{2}$, and 1.

We completely factor f as follows:

$$\begin{aligned} f(x) = 2x^3 + 11x^2 - 7x - 6 &= (x + 6)(2x^2 - x - 1) \\ &= (x + 6)(2x + 1)(x - 1) \end{aligned}$$

DISCRIMINANT OF A QUADRATIC EQUATION

The discriminant determines the number and type of solutions.

$b^2 - 4ac > 0$	2 real solutions
$b^2 - 4ac = 0$	1 real solution
$b^2 - 4ac < 0$	2 imaginary solutions