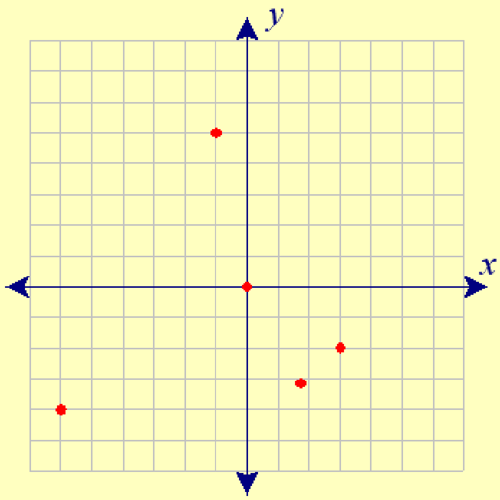
CHAPTER 2

RELATIONS AND FUNCTIONS

Relations, Domains, and Ranges

A **relation** is a set of ordered pairs. Any set of ordered pairs automatically relates the set of first coordinates to the set of second coordinates, and these sets have special names. The **domain** of a relation is the set of all the first coordinates, and the **range** of a relation is the set of all second coordinates.

a. The set $R = \{ (3, -2), (-1, 5), (-6, -4), (0, 0), (\sqrt{3}, -\pi) \}$ is a relation consisting of five ordered pairs. The domain of R is the set $\{ 3, -1, -6, 0, \sqrt{3} \}$, as these five numbers appear as first coordinates in the relation. The range of R is the set $\{ -2, 5, -4, 0, -\pi \}$, as these are the numbers that appear as second coordinates. The *graph* of this relation is simply a picture of the five ordered pairs plotted in the Cartesian plane, as shown below.



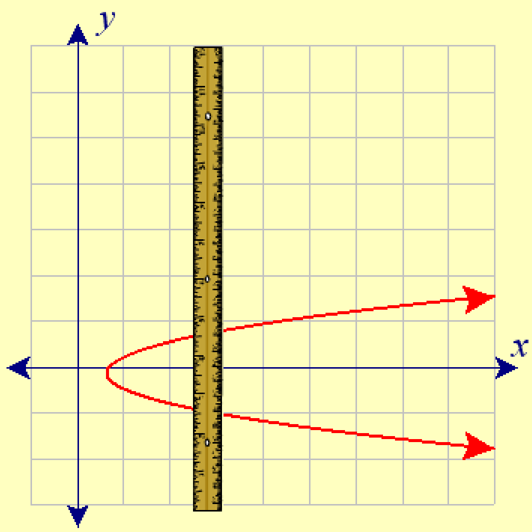
Function

A **function** is a relation in which every element of the domain is paired with *exactly one* element of the range. Equivalently, a function is a relation in which no two distinct ordered pairs have the same first coordinate.

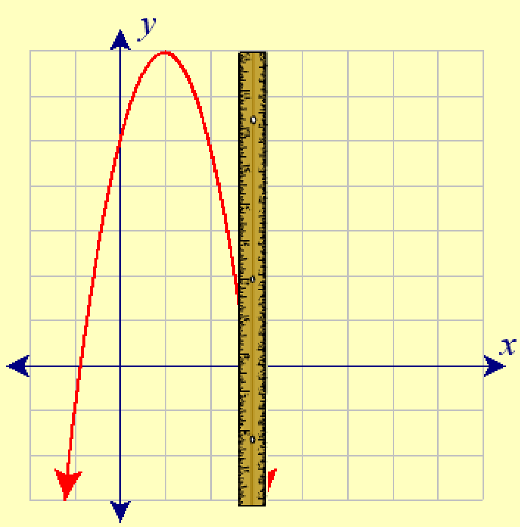
The Vertical Line Test

If a relation can be graphed in the Cartesian plane, the relation is a function if and only if no vertical line passes through the graph more than once. If even *one* vertical line intersects the graph of the relation two or more times, the relation fails to be a function.

a. The relation graphed below is not a function, as there are many vertical lines that intersect the graph more than once. The ruler is one such vertical line.



b. The relation graphed below is a function. In this case, every vertical line in the plane intersects the graph exactly once.



[Linear and Quadratic Functions]

Linear

Linear functions ||1st degree with one variable. Y=ax+b a≠0 || f(x)=ax+b a≠0

Linear Functions

A **linear function** f of one variable, say the variable x, is any function that can be written in the form $f(x) = ax + b$, where a and b are real numbers. If $a \neq 0$, a function $f(x) = ax + b$ is also called a **first-degree function**.

Show examples:

linear equation (first degree equations)

f(x) = -4x+2 || show the graph and talk about domain and range, do a vertical line test.

Slope is m= -4/1 and y intercept is (0,2) do graph

f(x) = (3+6x) / 3 || show graph and do domain , range, vertical line test.

The function can be rewritten in the form of 1+ 2x by using the least common multiple 3/3 and the problem can rewritten as y = 2x+1 whereby the slope is m= 2/1 and y intercept is (0,1)

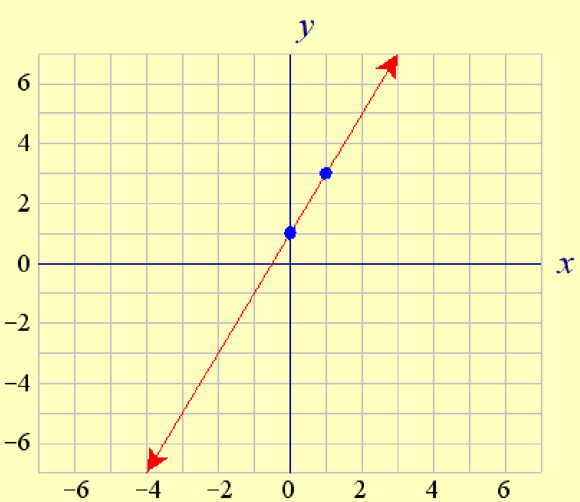
The domain will be all values for x for each and every point on this line because if a vertical line test is done it only hits the line once per point.

The range will be all values for y for each and every point on this line because if a vertical line test is done it only hits the line once per point.

Graph the following linear functions.

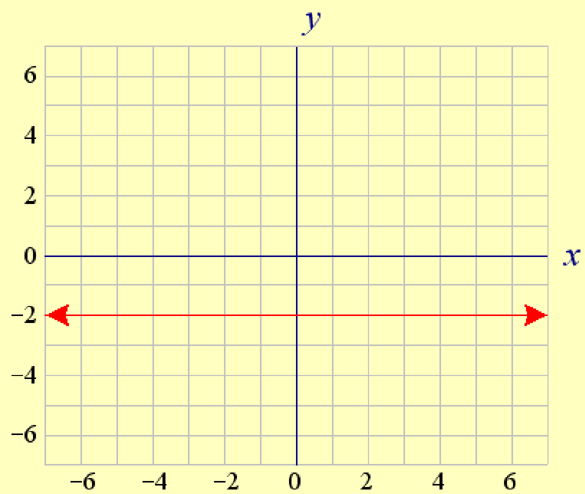
a. $f(x) = \dfrac{3+6x}{3}$ **b.** $g(x) = -2$

a.



The function f can be rewritten as $f(x) = 2x + 1$, and in this form we recognize it as a line with a slope of 2 and a y-intercept of 1. To graph this function, then, we can plot the y-intercept $(0, 1)$ and locate another point on the line by moving up 2 and over to the right 1 unit, giving us the ordered pair $(1, 3)$. Once these two points have been plotted, connecting them with a straight line completes the process.

b. $g(x) = -2$



The graph of the function g is a straight line with a slope of 0 and a y-intercept of -2. A linear function with a slope of 0 is also called a constant function, as it turns any input into one fixed constant, in this case the number -2. The graph of a constant function is always a horizontal line.

Quadratic

Quadratic functions ||2nd degree with two variables. Y= ax^2+bx+c a≠0 || f(x)=ax^2+bx+c a≠0

Show examples from the chap 2 quadratic equation

Quadratic Functions

A **quadratic, or second-degree, function** f of one variable, say the variable x, is any function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

x	$f(x)$
–3	9
–1	1
0	0
2	4

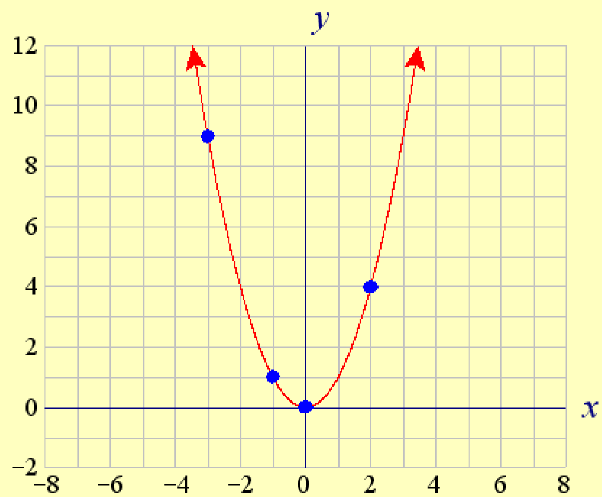
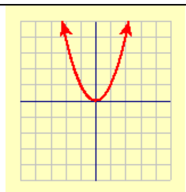
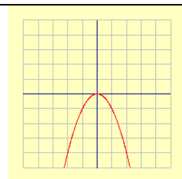
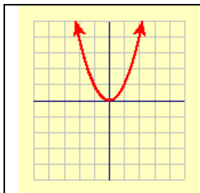
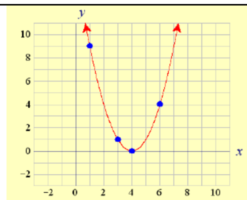


Figure 2: Graph of $f(x) = x^2$

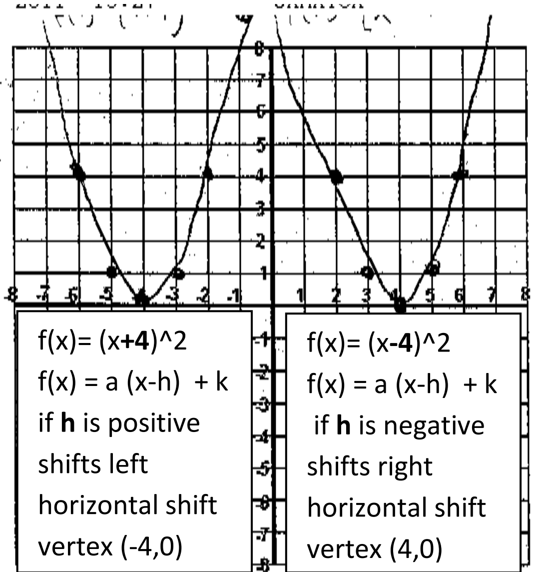
f(x)= x^2		f(x) = -x^2		Talk about parabola, symmetric over axis.
Vertex (0,0) – in a positive parabola it is the lowest point and			Vertex (0,0) - the highest point if it is a negative parabola.	

<table>
<tr>
<td>
f(x)= x^2</td>
<td>
f(x)= (x-4)^2</td>
<td><table><tr><th>x</th><th>g (x)</th></tr><tr><td>1</td><td>9</td></tr><tr><td>3</td><td>1</td></tr><tr><td>4</td><td>0</td></tr><tr><td>6</td><td>4</td></tr></table>Vertex (4, 0)</td>
<td>Vertex form of a Quadratic Function
f(x) = a (x-h)^2 +k
a || if a > 1 = narrower or skinnier ||
a ||if 0<a<1 = broader or fatter
h || shifts the function (+) left or (-) right depending on the sign.
k || shifts the function up or down depending on the sign.</td>
</tr>
<tr>
<td>Vertex (0,0)</td>
<td>Vertex (4,0)</td>
<td></td>
<td>DO the f(x)= (x-4)^2 from the graph paper done.</td>
</tr>
</table>

In order to shift the parabola horizontally on the x-axis
we add a positive value to the existing term to move it to the **left**.
|| f (x) = (x+4)^2 || thus changing the vertex from (0,0) to (-4,0)

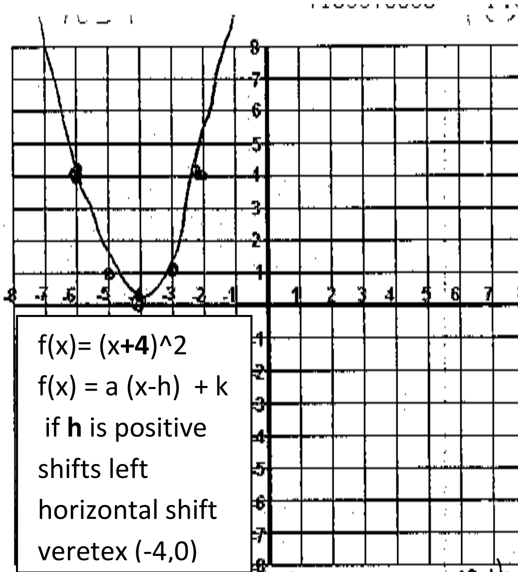
we add a negative value to the existing term to move it to the **right**.
|| f (x) = (x-4)^2 || thus changing the vertex from (0,0) to (4,0)

x	f(x)
0	16
2	4
3	1
4	0
5	1
6	4
7	9
8	16



f(x)= (x**+4**)^2
f(x) = a (x-h) + k
if **h** is positive
shifts left
horizontal shift
vertex (-4,0)

f(x)= (x**-4**)^2
f(x) = a (x-h) + k
if **h** is negative
shifts right
horizontal shift
vertex (4,0)



f(x)= (x**+4**)^2
f(x) = a (x-h) + k
if **h** is positive
shifts left
horizontal shift
veretex (-4,0)

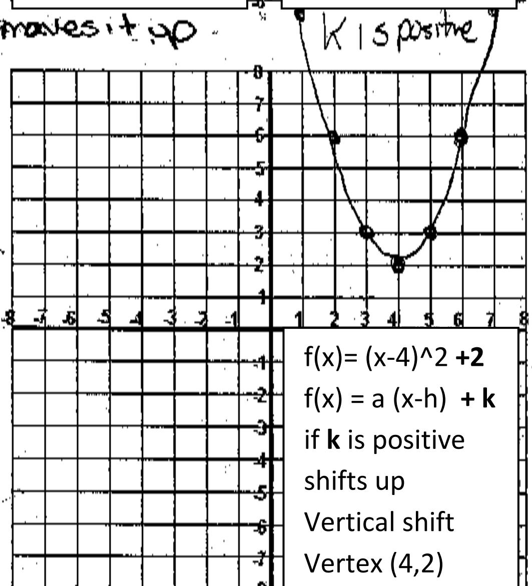
x	f(x)
-1	9
-2	4
-3	1
-4	0
-5	1
-6	4
-7	9

changing the sign moves it left to right

f(x) = (x-4)^2 moves it up

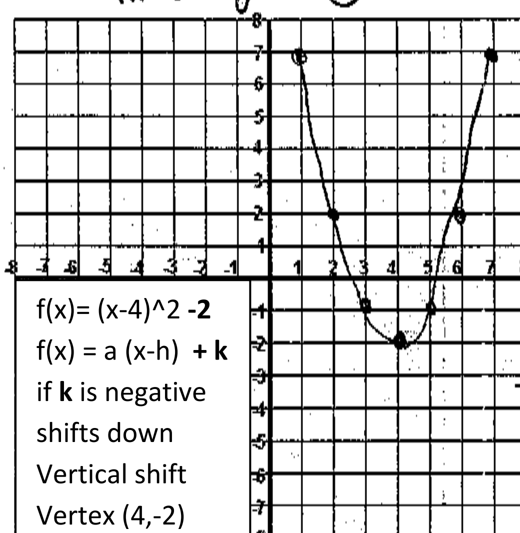
x	f(x)+2
1	11
2	6
3	3
4	2 (vertex)
5	3
6	6
7	11
8	18

k is positive



f(x)= (x-4)^2 **+2**
f(x) = a (x-h) **+ k**
if **k** is positive
shifts up
Vertical shift
Vertex (4,2)

k is negative — moves it down



f(x)= (x-4)^2 **-2**
f(x) = a (x-h) **+ k**
if **k** is negative
shifts down
Vertical shift
Vertex (4,-2)

x	f(x)-2
1	7
2	2
3	-1
4	-2
5	-1
6	2
7	7
8	14

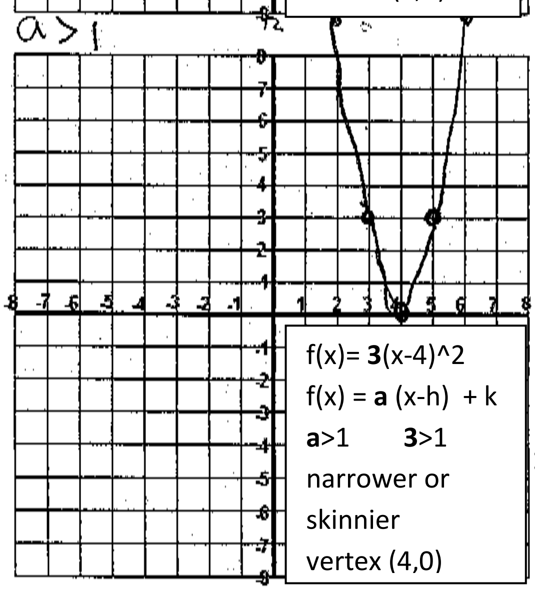
Broader between 0 and 1

To make it skinny a(x)^2

3(x-4)^2

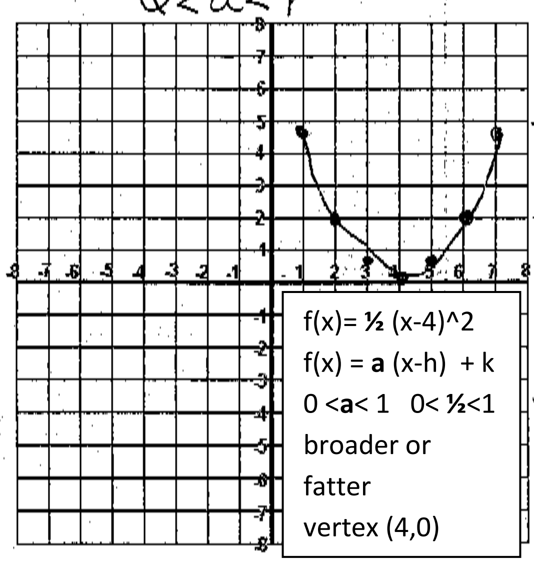
a > 1

x	f(x)
2	12
3	3
4	0
5	3
6	12



f(x)= **3**(x-4)^2
f(x) = **a** (x-h) + k
a>1 **3**>1
narrower or
skinnier
vertex (4,0)

0 < a < 1

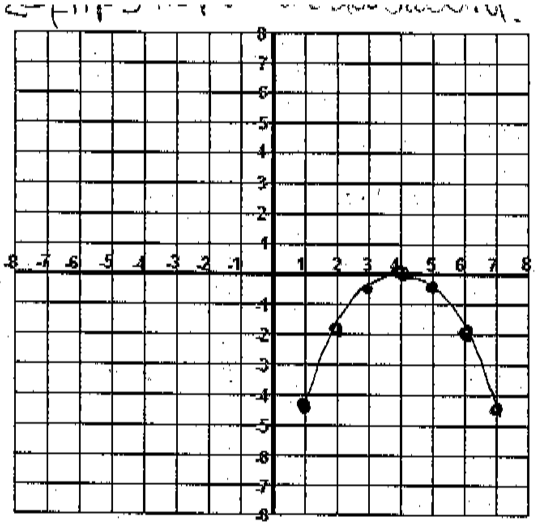


f(x)= ½ (x-4)^2
f(x) = **a** (x-h) + k
0 <**a**< 1 0< ½<1
broader or
fatter
vertex (4,0)

.5(x-4)^2

x	f(x)
1	4.5
2	2
3	.5
4	0
5	.5
6	2
7	4.5

x	f(x)
1	-4.5
2	-2
3	-.5
4	0
5	-.5
6	-2
7	-4.5



f(x)= - ½ (x-4)^2

f(x) = - **a** (x-h) + k

0 <**a**< 1 0< ½<1

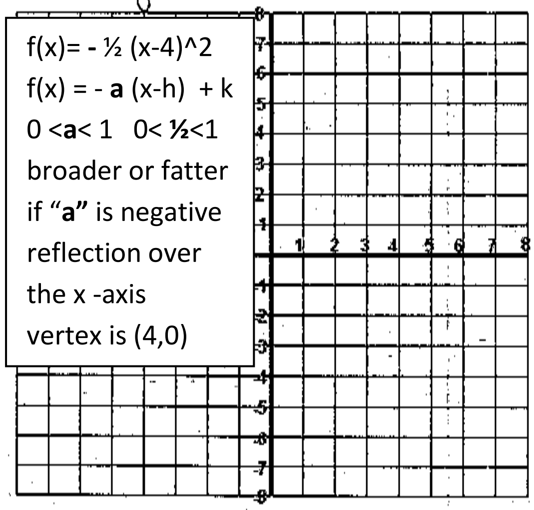
broader or fatter

if "**a**" is negative

reflection over

the x -axis

vertex is (4,0)



The basic function of

f(x) = (x-4)^2

is the f(x) = x^2

the graph to the right is **f(x) = x^2**

vertex is (0,0)

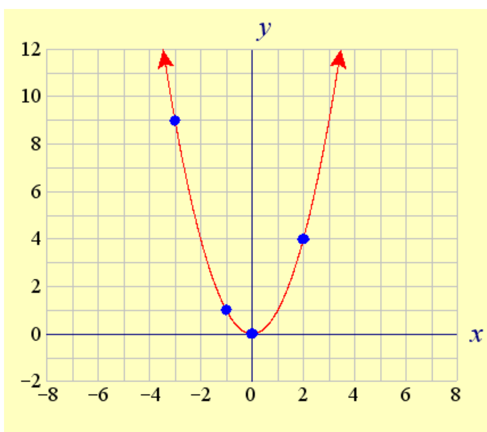
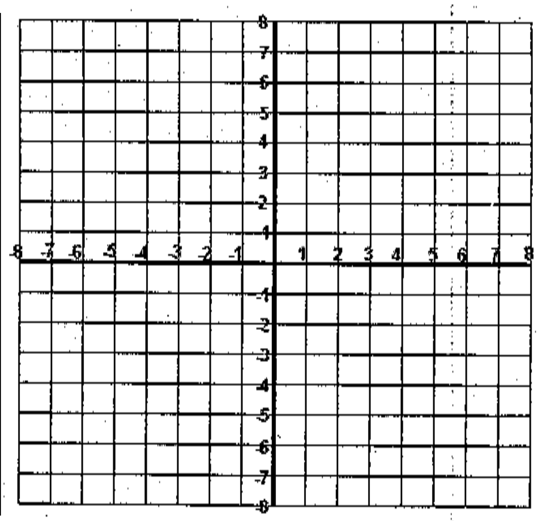
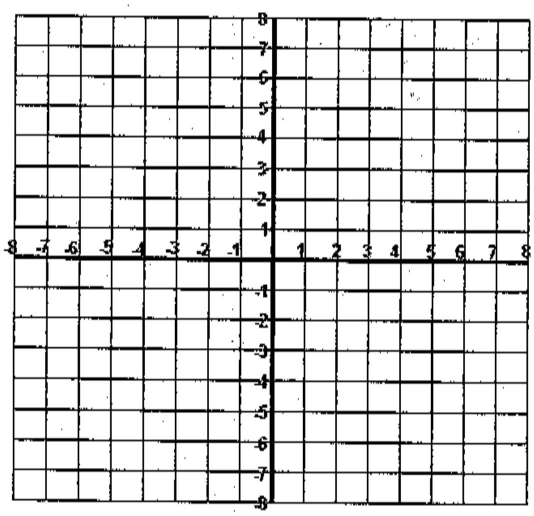
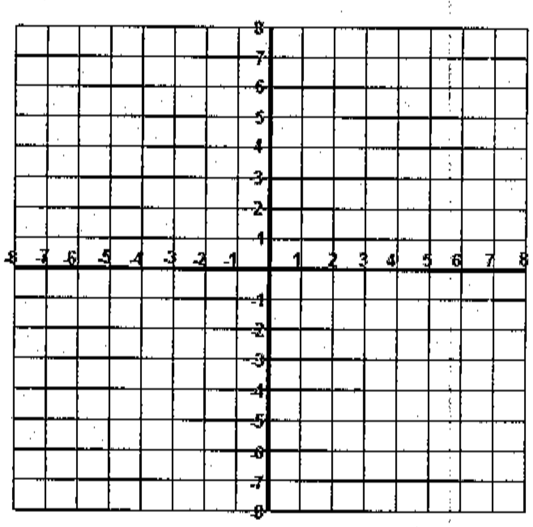


Figure 2: Graph of $f(x) = x^2$







Problems using vertex form of a quadratic equation. f(x) = a (x-h)^2 +k f(x)=ax^2+bx+c a≠0f(x)

(1) Find the vertex of this function: f(x) = -x^2-2x+8

$$f(x) = -x^2 - 2x + 8$$

$$f(x) - 1 = -(x^2 + 2x) + 8$$

f(x) -1 (1) = -(x^2 +2x +1) +8

f(x) -1 = -(x^2 +2x +1) +8

+1 = +1

f(x) 0 = - (x^2 +2x +1)+1+8

f(x)= - (x+1) (x+1) +9

f(x)= - (x+1)^2 +9

Vertex = (-1,9)

Step1
We place -1 to balance the left of the equal sign of the equation to compensate for the -1 taken out.
Complete the square

Step2
We place a positive 1 to the left of the equal sign of the equation due to completing the square.

Step3
Multiply -1 to 1 = -1 and then add +1 to the left and right of the equal sign to eliminate the left side and make it equal to zero.

Step4
The left is now equal to 0 and the right will add 1 to 8 thus getting 9

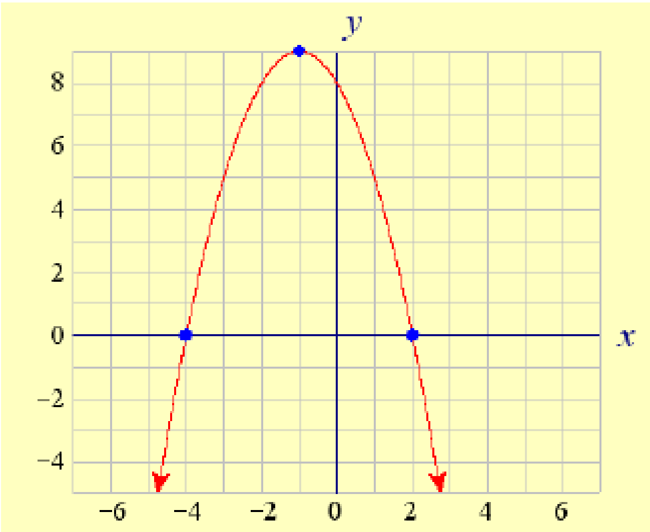
Vertex = (-1,9)

(2) Determine the number of x-intercept(s), then enter the x-intercept(s), if any, of this function as ordered pair(s) below.

-x^2-2x+8 = 0 || ax^2+bx+c =0 to find x intercepts if any
-x^2-2x+8 = 0 || Change the sign of all the terms
x^2+2x-8 = 0 || Factor quadratic
(x+4)(x-2)= 0 ||
(x+4)=0 (x-2)= 0 || find x intercepts
x=-4 x=2 || x intercepts

(3) Graph this quadratic function by identifying two points on the parabola other than the vertex and the x-intercepts.

x	**f(x) =**	**PLOT**
-4	**0**	**(-4,0) x-intercept**
-3	**5**	**(-3,5)**
-2	**8**	**(-2,8)**
-1	**9**	**(-1,9) vertex**
0	**8**	**(0,8)**
1	**5**	**(1,5)**
2	**0**	**(2,0) x-intercept**



Finding minimums and maximums and testing for vertex f(x)=ax^2+bx+c a≠0f(x)

a < 0 then f(-(b/2a)) is the max point // a > 0 then f(-(b/2a)) is the min point vertex (x,y) vertex ((-(b/2a)),f(-(b/2a)))

f(x) = -x^2-2x+8 // a= -1 b= -2 c= 8
a<0 the graphs vertex will its max point. Thus f(x) = -1 and it is also the x value of the vertex so solve -x^2-2x+8 to find the y value of the vertex. -(-1)^2 – 2(-1) +8 = 9 Thus the vertex is (-1,9)

Consider the following quadratic function.

$$t(x) = -(x+5)^2 - 3$$

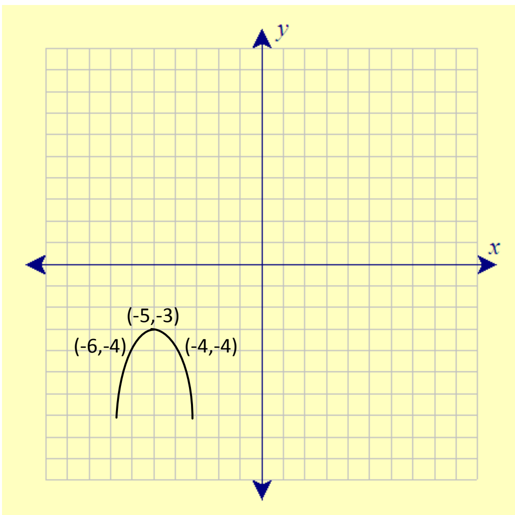
Step 1. Find the vertex of this function.

Step 1: (−5, −3)

Step 2. Determine the number of x-intercept(s), then enter the x-intercept(s), if any, of this function as ordered pair(s) below.

Step 2: None, the function does not touch the x-axis.

Step 3. Graph this quadratic function by identifying two points on the parabola other than the vertex and the x-intercepts.



A:

B:

Points
(-6,-4)
(-4,-4)
(-7,-7)
(-3,-7)

Step 3: A= (−6, −4) , B= (−4, −4)

x	f(x) = y	$-(x+5)^2$ -3	PLOT
-7	-7	-((-7)+5)^2 -3 = -(-2)^2 – 3 = -4-3 = **-7**	**(-7-7)**
-6	-4		**(-6,-4)**
-5	**-3**		**(-5,-3) vertex**
-4	-4		**(-4,-4)**
-3	-7		**(-3,-7)**

Complete the square of the given quadratic expression. Then, graph the function using the technique of shifting.

$f(x) = x^2 - 10x + 19$

Complete the square.

$x^2 - 10x + 19 = 0$	Set the function equal to 0.
$x^2 - 10x = -19$	Subtract 19 from both sides.
$x^2 - 10x + \left(\frac{-10}{2}\right)^2 = -19 + \left(\frac{-10}{2}\right)^2$	Complete the square.
$x^2 - 10x + 25 = -19 + 25$	Simplify both sides.
$(x-5)^2 = 6$	Factor the perfect square trinomial.
$(x-5)^2 - 6 = 0$	Subtract 6 from both sides to get all terms on the left side.

The function is now $f(x) = (x-5)^2 - 6$.

VERTEX = (5,-6)   f(x) = a (x-h)^2 +k

Now, graph the function. Start with the graph of the basic function $y = x^2$.

Compare the functions $y = x^2$ and $y = (x-5)^2 - 6$. The difference between the two functions is that, in the latter one, 5 is subtracted from the argument x and 6 is subtracted from the right side of the function, the value of y.

Thus, the graph of $y = (x-5)^2 - 6$ is the same as the graph of $y = x^2$ with a horizontal shift to the right and a vertical shift down.

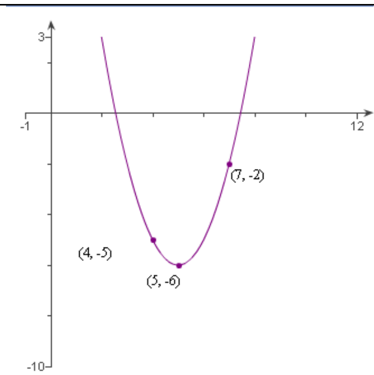
For each point (x,y) on the graph of $y = x^2$, add 5 to the x-value and subtract 6 from the y-value. The resulting ordered pair will be a point on the graph of $y = (x-5)^2 - 6$.

The point $(-1,1)$ is on the graph of the function $y = x^2$. So, the point $(-1+5, 1-6) = (4,-5)$ must be on the graph of the function $y = (x-5)^2 - 6$.

The point $(0,0)$ is on the graph of the function $y = x^2$. So, the point $(0+5, 0-6) = (5,-6)$ must be on the graph of the function $y = (x-5)^2 - 6$.

The point $(2,4)$ is on the graph of the function $y = x^2$. So, the point $(2+5, 4-6) = (7,-2)$ must be on the graph of the function $y = (x-5)^2 - 6$.

Plot the points $(4,-5)$, $(5,-6)$, and $(7,-2)$ and draw a curve through them to form the graph of $f(x) = x^2 - 10x + 19$.



Complete the square of the given quadratic expression. Then, graph the function using the technique of shifting.

$f(x) = -2x^2 - 20x - 43$

In preparation for completing the square, we will factor -2 out of the x-terms as follows.

$$f(x) = -2x^2 - 20x - 43$$
$$f(x) = -2(x^2 + 10x) - 43$$

To complete the square of $x^2 + mx$, we must add $\left(\frac{m}{2}\right)^2$.

In the equation $f(x) = -2(x^2 + 10x) - 43$, the value of m is 10.

When m is 10, $\left(\frac{m}{2}\right)^2$ will be 25.

Complete the square by entering the appropriate numbers to add and subtract in the expression below.

$$f(x) = -2(x^2 + 10x) - 43$$
$$f(x) = -2(x^2 + 10x + 25) - 43 + 50$$

f(x) -2 = -2 (x^2+10x) – 43 [add -2 to the left of the =]
f(x) -2 (+25) = -2 (x^2+10x+25) – 43 [add +25 to the left of the =]
f(x) -50 = -2 (x^2+10x+25) – 43 [multiply -2 *25=-50]
+50 = +50
f(x) 0 = -2 (x^2+10x+25) – 43 +50 [bring to the right side -50 → 50]
f(x) 0= -2 (x+5)^2 +7 [subtracting -43 from 50 = +7]
f(x) = -2 (x+5)^2 +7 → vertex is (-5,7)

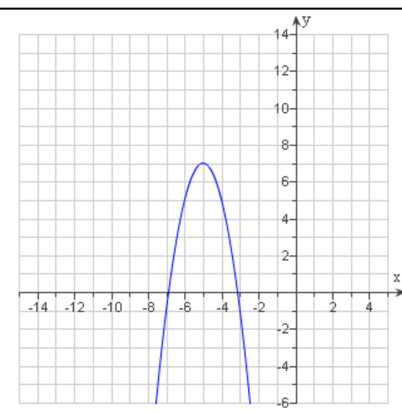
Simplify.

$$f(x) = -2(x^2 + 10x + 25) - 43 + 50$$
$$f(x) = -2(x^2 + 10x + 25) + 7$$

Because the polynomial $x^2 + 10x + 25$ is square, we can factor it as follows.

$$f(x) = -2(x^2 + 10x + 25) + 7$$
$$f(x) = -2(x + 5)^2 + 7$$

Factor and get -2(x+5)(x+5)+7



To graph $f(x) = -2(x + 5)^2 + 7$, we should transform the function $f(x) = x^2$.

To transform $f(x) = x^2$ into $f(x) = -x^2$, we must reflect about the x-axis.

To transform $f(x) = -x^2$ into $f(x) = -2x^2$, we must stretch vertically by a factor of 2.

To transform $f(x) = -2x^2$ into $f(x) = -2(x + 5)^2$, we must shift 5 units to the left.

To transform $f(x) = -2(x + 5)^2$ into $f(x) = -2(x + 5)^2 + 7$, we must shift 7 units up.

See page 11 problem 33 of chapter 3 review. q(x) = -2x^2 +4x

(1)Find the vertex

(1,2) test using

a < 0 then f(-(b/2a)) is the max point // a > 0 then f(-(b/2a)) is the min point vertex (x,y) vertex ((-(b/2a)),f(-(b/2a)))

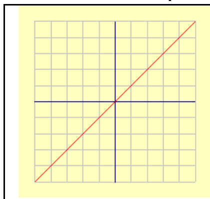
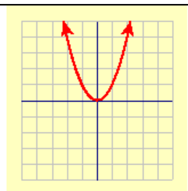
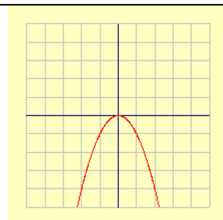
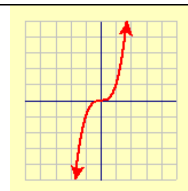
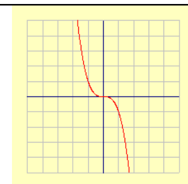
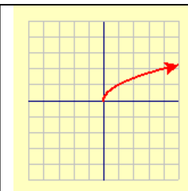
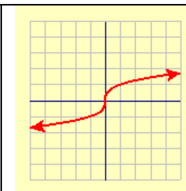
(2)Determine x intercepts

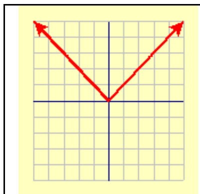
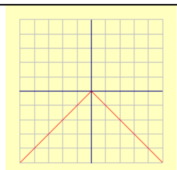
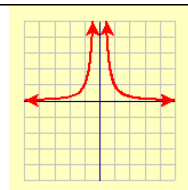
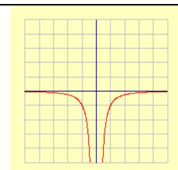
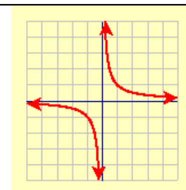
x	f(x) =	-2x^2 +4x	PLOT
-1	-6		(-,5)
0	0	-2(0)^2+ 4(0)= 0	(0,0) x-intercept
1	2		(1,2) vertex
2	0		(2,0) x-intercept
3	-6		(1,5)

-2x^2 +4x // -2x (x-2) // -2x=0 thus x=0 // x-2 = 0 thus x=2 // x intercepts are (0,0) & (2,0)

(3)graph

Quadratic Equations and other common functions

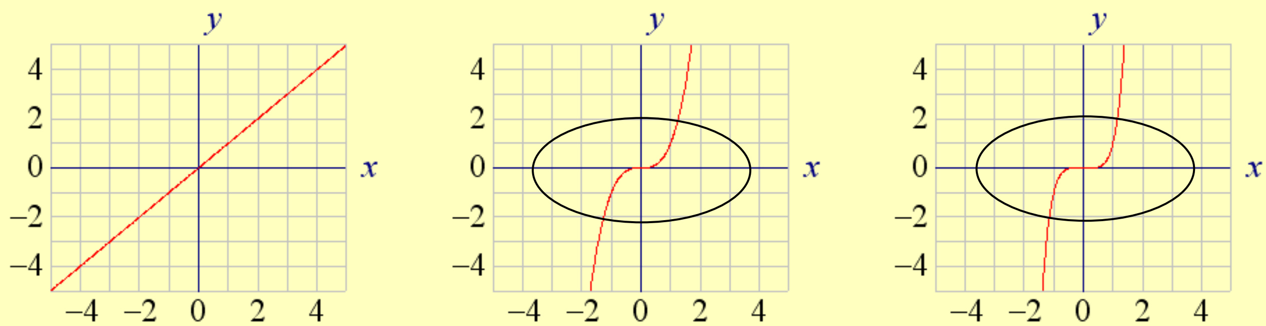
						
f(x)= x or x^1 positive odd	f(x)= x^2 positive even	f(x)= -x^2 negative even	f(x)= x^3 odd positive	f(x)= -x^3 odd negative	f(x)= x^1/2 even	f(x)= x^1/3 odd

						
f(x)= |x|	f(x)= -2 |x|	f(x)= 1/x^2 positive even	f(x)= - (1/x^2) negative even	f(x)= 1/x^1 odd positive	f(x)= - (1/x^1) negative odd	f(x)=

Other functions

Functions of the Form ax^n

Consider the graphs in Figure 1:



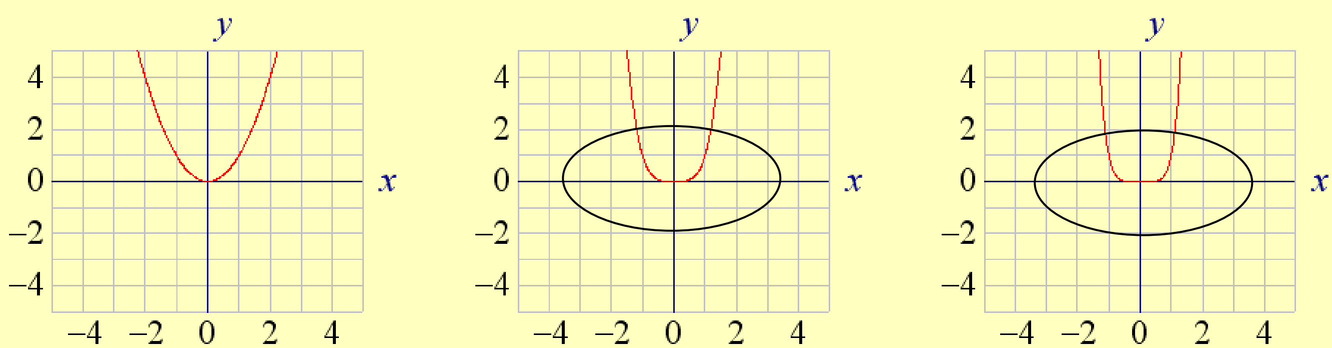
The function $f(x) = x$ The function $f(x) = x^3$ The function $f(x) = x^5$

Figure 1: Odd Exponents

The three graphs in Figure 1 show the behavior of $f(x) = x^n$ for the first three odd exponents. Note that in each case, the domain and the range of the function are both the entire set of real numbers; the same is true for higher odd exponents as well.

The higher the degree the flatter the function gets in relation to the x axis.

Now, consider the graphs in Figure 2:



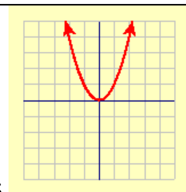
The function $f(x) = x^2$ The function $f(x) = x^4$ The function $f(x) = x^6$

Figure 2: Even Exponents

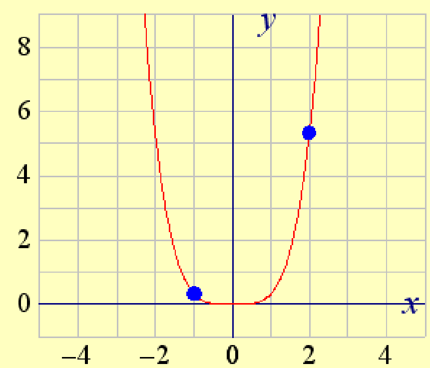
The higher the degree the flatter the function gets in relation to the x axis.

$$f(x) = ax^n.$$

a | | if a > 1 = narrower or skinnier | | a | |if 0<a<1 = broader or fatter

a. $f(x) = \dfrac{x^4}{3}$	Original graph is  = x^2

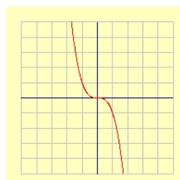
a.

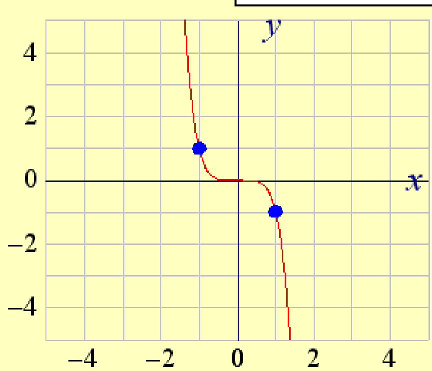


The graph of this function will have the same basic shape as the function x^4, but compressed a bit vertically because of the factor of $\frac{1}{3}$. To make our sketch reasonably accurate, we can calculate the coordinates of a few points on the graph. The graph to the left illustrates that $f(-1) = \frac{1}{3}$ and $f(2) = \frac{16}{3}$.

b. $f(x) = -x^5$

Solution:

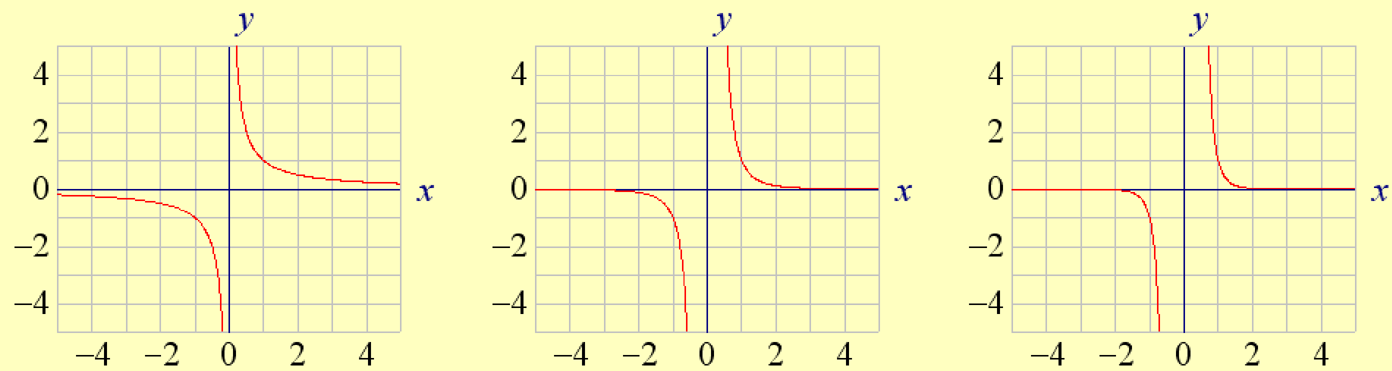
Original function graph is  **= - x^3**



We know that the function f will have the same shape as the function x^5, but reflected with respect to the x-axis because of the factor of -1. The graph to the left illustrates this point.

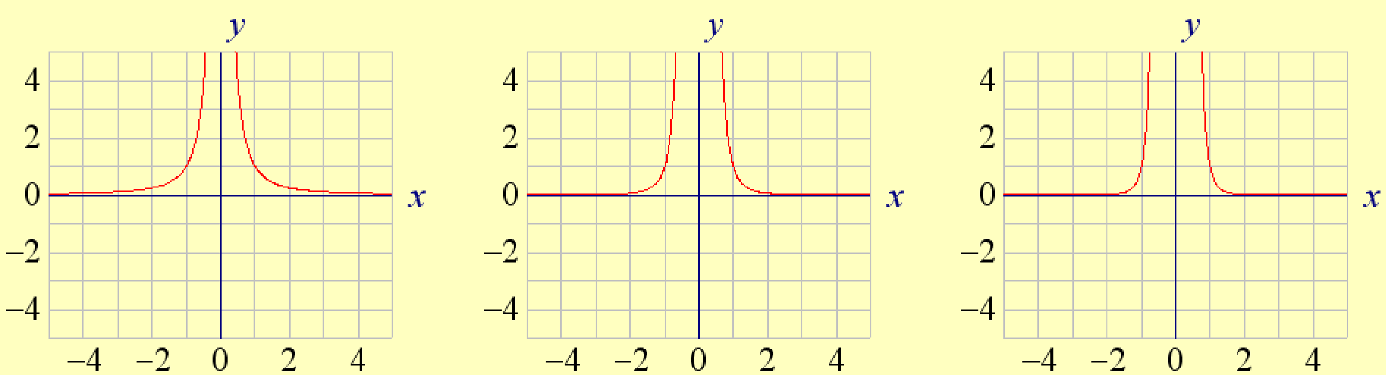
We have plotted a few points on the graph of f, namely $(-1, 1)$ and $(1, -1)$, just to verify our analysis.

Functions of the Form $\frac{a}{x^n}$ or ax^{-n}



The function $f(x) = \frac{1}{x}$ The function $f(x) = \frac{1}{x^3}$ The function $f(x) = \frac{1}{x^5}$

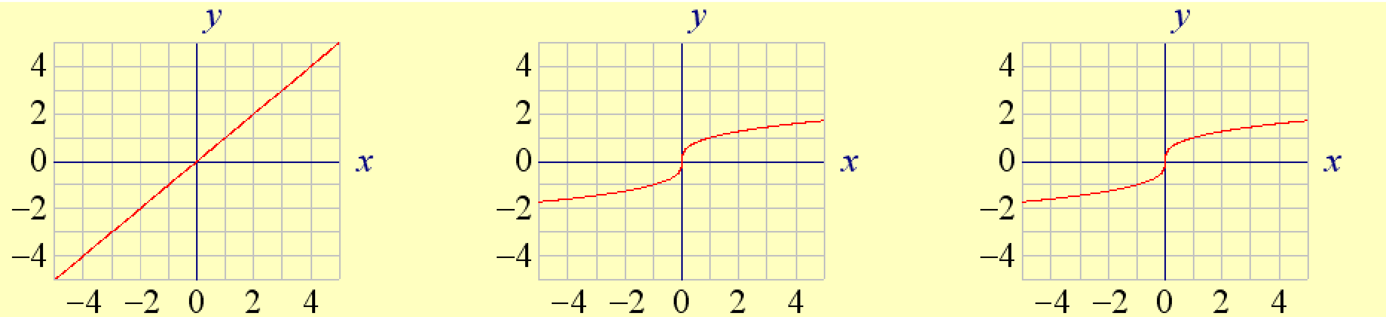
Figure 3: Odd Exponents



The function $f(x) = \frac{1}{x^2}$ The function $f(x) = \frac{1}{x^4}$ The function $f(x) = \frac{1}{x^6}$

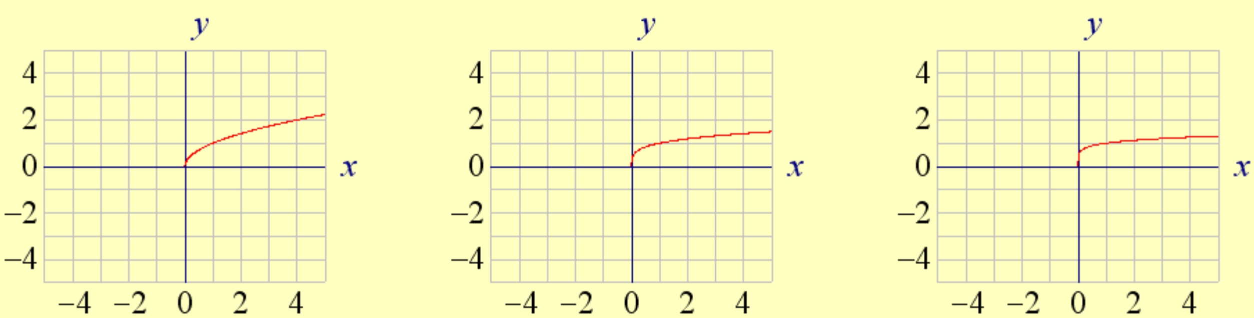
Figure 4: Even Exponents

Functions of the Form $ax^{\frac{1}{n}}$ or $a\sqrt[n]{x}$



The function $f(x) = x$ The function $f(x) = x^{\frac{1}{3}}$ The function $f(x) = x^{\frac{1}{5}}$

Figure 5: Odd Roots



The function $f(x) = x^{\frac{1}{2}}$ The function $f(x) = x^{\frac{1}{4}}$ The function $f(x) = x^{\frac{1}{6}}$

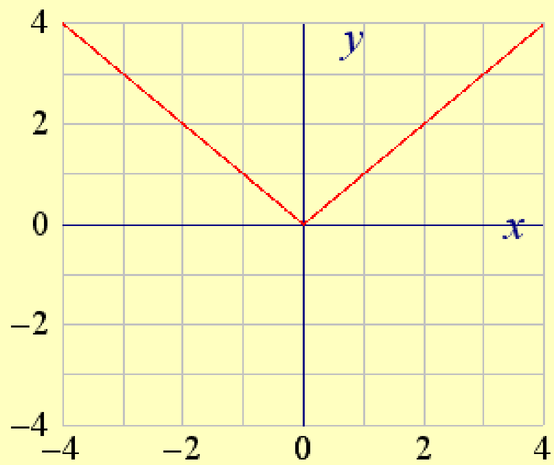
Figure 6: Even Roots

The Absolute Value Function $f(x) = |\ x\ |$

$$|\ x\ | = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Figure 7:

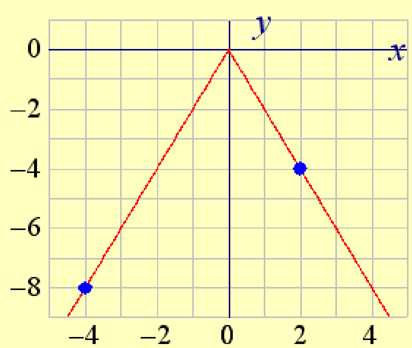
The Absolute Value Function



Sketch the graph of the function $f(x) = -2|\ x\ |$.

Solution:

The graph of f will be a stretched version of $|\ x\ |$, reflected with respect to the x-axis. As always, we can plot a few points to verify that our reasoning is correct. In the graph below, we have plotted the values of $f(-4)$ and $f(2)$.



Horizontal Shifting

Let $f(x)$ be a function whose graph is known, and let h be a fixed real number. If we replace x in the definition of f by $x - h$, we obtain a new function $g(x) = f(x - h)$. The graph of g is the same shape as the graph of f, but shifted to the right by $|h|$ units if $h > 0$ and shifted to the left by $|h|$ units if $h < 0$.

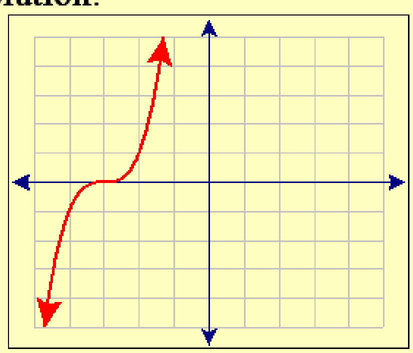
Caution!

It is easy to forget that the negative sign in the expression $x - h$ is critical. It may help to remember a few specific examples: replacing x with $x - 5$ shifts the graph 5 units to the *right*, since 5 is positive. Replacing x with $x + 5$ shifts the graph 5 units to the *left*, since we have actually replaced x with $x - (-5)$. With practice, the effect of replacing x with $x - h$ on the graph of a function will come to seem natural.

a. $f(x) = (x + 3)^3$

Solution:

a.



The shape of $(x+3)^3$ is the same as the shape of the graph of x^3, since our expression is obtained from the other by replacing x with $x + 3$. We simply draw the basic cubic shape (the shape of x^3) shifted to the left by 3 units. Note, for example, that $(-3, 0)$ is one point on the graph.

Vertical Shifting

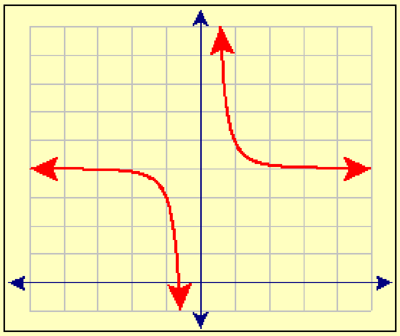
Let $f(x)$ be a function whose graph is known, and let k be a fixed real number. The graph of the function $g(x) = f(x) + k$ is the same shape as the graph of f, but shifted upward $|k|$ units if $k > 0$ and shifted downward $|k|$ units if $k < 0$.

Sketch the graphs of the following functions.

a. $f(x) = \dfrac{1}{x^3} + 4$ **b.** $g(x) = \sqrt[5]{x} - 3$

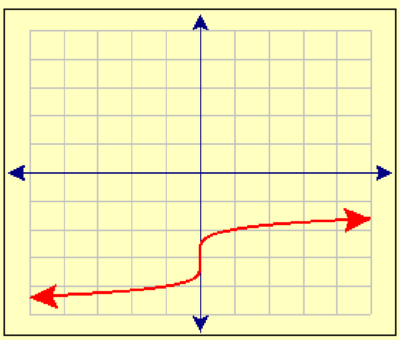
Solution:

a.



The graph of $f(x) = \dfrac{1}{x^3} + 4$ is the graph of $\dfrac{1}{x^3}$ shifted up 4 units. Note that this doesn't affect the domain: the domain of f is $(-\infty, 0) \cup (0, \infty)$, the same as the domain of $\dfrac{1}{x^3}$. However, the range is affected. The range of f is $(-\infty, 4) \cup (4, \infty)$.

b.



To graph $g(x) = \sqrt[5]{x} - 3$, we simply shift the graph of $\sqrt[5]{x}$ down by 3 units.

Reflecting with Respect to the Axes

Let $f(x)$ be a function whose graph is known.

1) The graph of the function $g(x) = -f(x)$ is the reflection of the graph of f with respect to the x-axis.

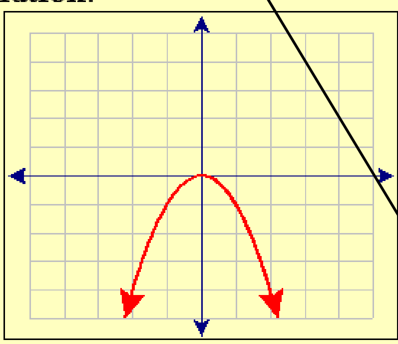
2) The graph of the function $g(x) = f(-x)$ is the reflection of the graph of f with respect to the y-axis.

Sketch the graphs of the following functions.

a. $f(x) = -x^2$ **b.** $g(x) = \sqrt{-x}$

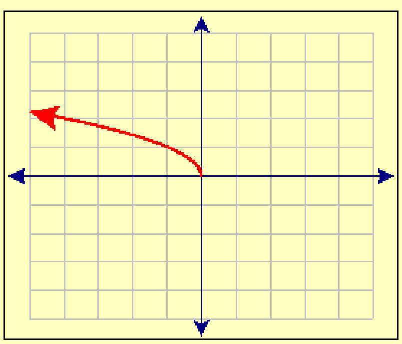
Solution:

a.



To graph $f(x) = -x^2$, we simply turn the graph of the prototypical parabola x^2 upside down. Note that the domain is still the entire real line, but the range of f is the interval $(-\infty, 0]$.

b.



To graph $g(x) = \sqrt{-x}$, we reflect the graph of $\sqrt{x}$ with respect to the y-axis. Note that this changes the domain, but not the range. The domain of g is the interval $(-\infty, 0]$ and the range is $[0, \infty)$.

Stretching and Compressing

Let $f(x)$ be a function whose graph is known, and let a be a positive real number.

1) The graph of the function $g(x) = af(x)$ is stretched vertically compared to the graph of f if $a > 1$.

2) The graph of the function $g(x) = af(x)$ is compressed vertically compared to the graph of f if $0 < a < 1$.

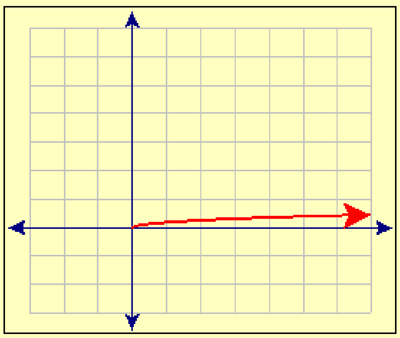
Sketch the graphs of the following functions.

a. $f(x) = \frac{\sqrt{x}}{6}$

b). g(x) = 4 | x |

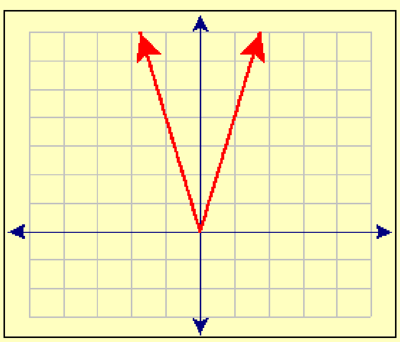
Solution:

a.



The graph of the function f has been compressed considerably, because of the factor of $\frac{1}{6}$. The shape is similar to the shape of $\sqrt{x}$, but all of the second coordinates have been multiplied by the factor of $\frac{1}{6}$, and are consequently smaller.

b.



In contrast to the last example, the graph of $g(x) = 4|x|$ is stretched compared to the standard absolute value function. Every second coordinate has been multiplied by a factor of 4, and is consequently larger.

Order of Transformations

If a function g has been obtained from a simpler function f through a number of transformations, g can usually most easily be understood by looking for the transformations in this order:

1. Horizontal shifts.
2. Stretching and compressing.
3. Reflecting.
4. Vertical shifts.

$g(x) = -2\sqrt{x+1} + 3.$

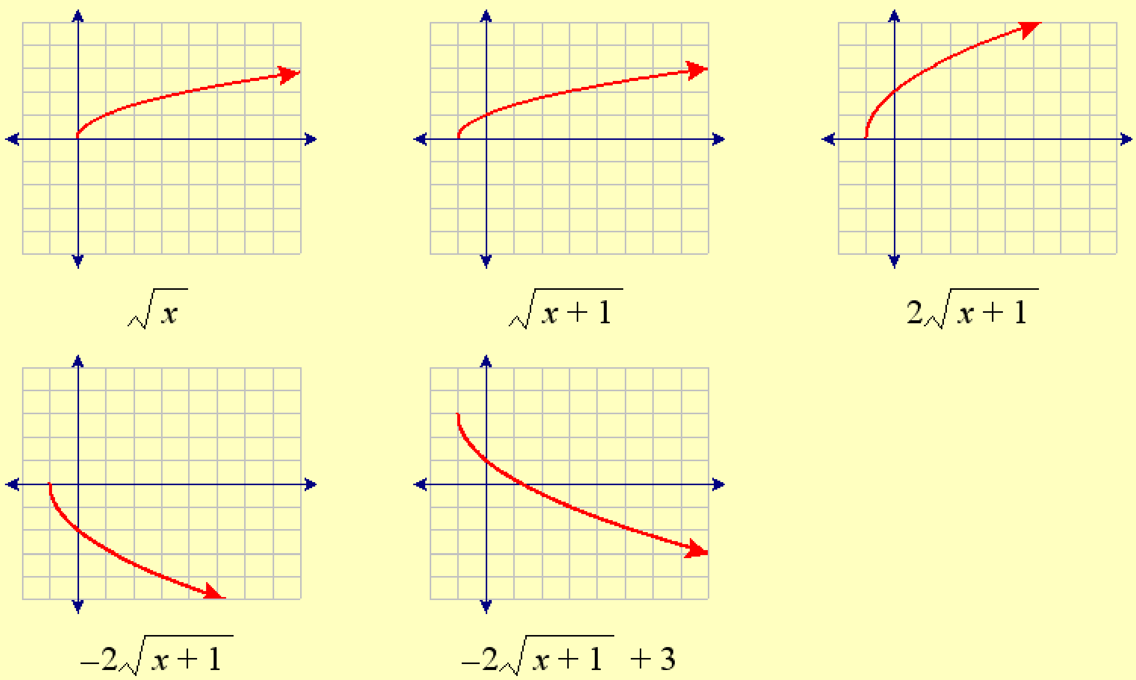
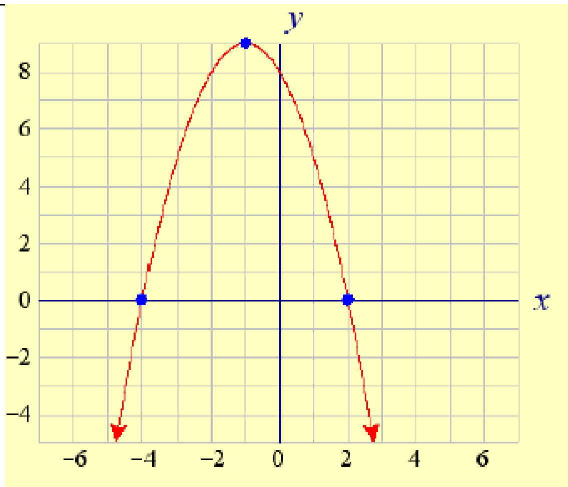


Figure 1: Building the Graph of $g(x) = -2\sqrt{x+1} + 3$

Determine the quadratic function whose graph is given the vertex is (-1,9) and y- intercept is (0,8)



The basic function of this graph is -x^2,

Next the quadratic should look in the form of

f(x) = a (x-h)^2 +k

We know the vertex is (h,k) → (-1,9)

Horizontal movement

f(x) = a (x-**h**)^2 +k

When “**h**” is positive the equation moves to the left and when “**h**” is negative the equation moves to the right.

Since the graph shifted to the left and the value for “**x**” in the vertex is -1 then “h” is positive and the value for “h” is 1. [x-h=0 thus -1+1 = 0]

Vertical Movement

f(x) = a (x-h)^2 +**k**

When “**k**” is positive the function moves up, and when “**k**” is negative the function moves down.

Since the graph shifted up “**k**” is positive.

Reflection over the x-axis

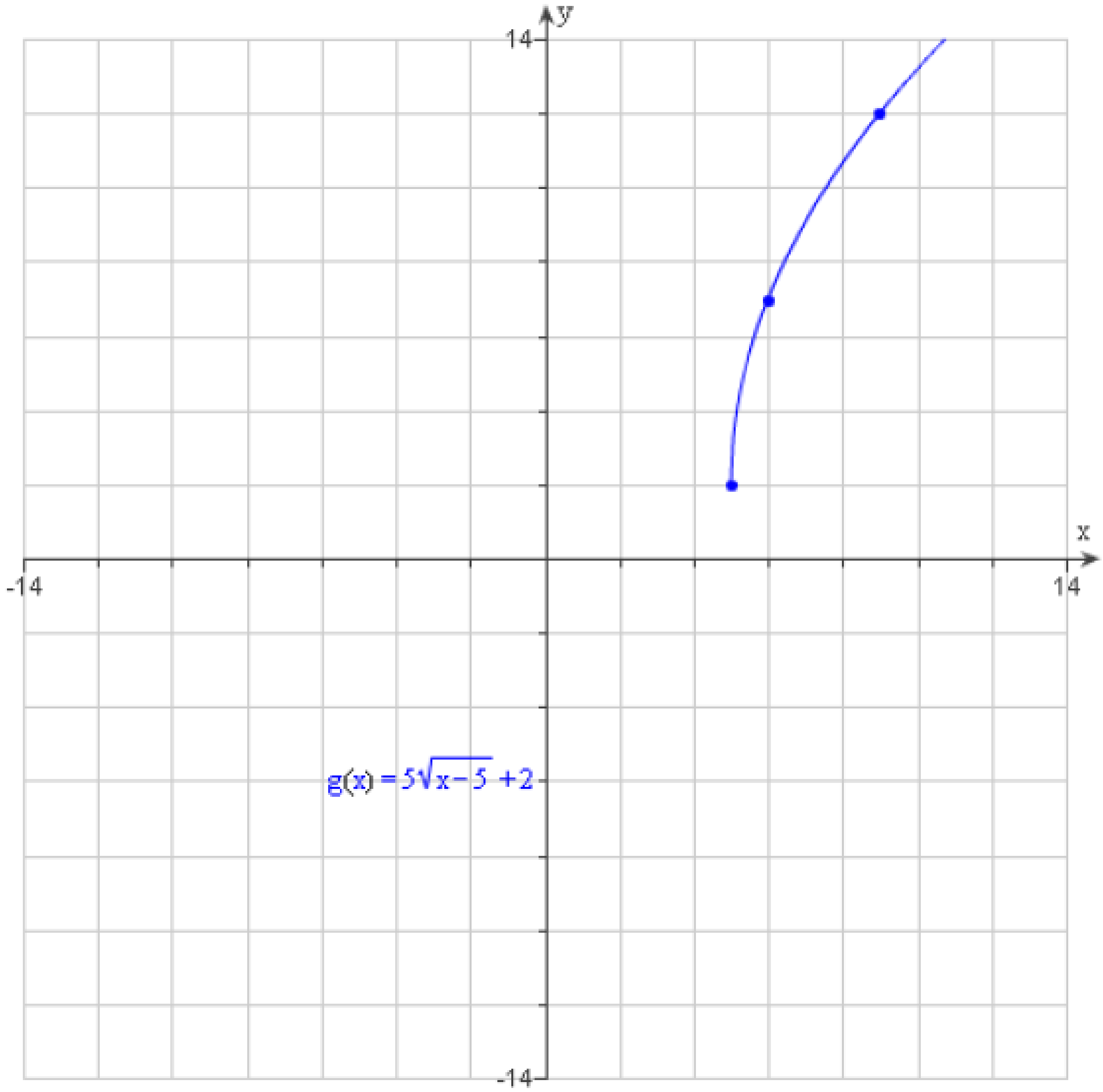
f(x) = **a** (x-h)^2 + k

When ever “**a**” is negative thus “**-a**”the function is reflected over the x-axis
If “**a**” is omitted then just keep the negative sign.

The formula will change from f(x) = a (x-**h**)^2 **+k** to f(x) = **-a** (x**+1**)^2 **+9** → f(x) = **-**(x**+1**)^2 **+9**

To test for the y-intercept replace “x” with 0 → f(0) = **-**(0**+1**)^2 **+9** → -1 +9 = 8 thus **(0,8)**

Domain and Range. Graphs and Functions. 2.5



find the domain and range of g(x).

The domain of g is the largest set of real numbers for which the value of g(x) is a real number.

Remember that the square root function is undefined for numbers less than zero. The function $g(x) = 5\sqrt{x-5} + 2$ is defined for all real numbers greater than or equal to 5. Therefore, the domain of $g(x) = 5\sqrt{x-5} + 2$ is $[5,\infty)$.

The range of g is the set of g(x)-values of the function that are images of the x-values in the domain.

The smallest value of $g(x) = 5\sqrt{x-5} + 2$ occurs at the point (5,2) and the values of g(x) continue to infinity. Therefore, the range of $g(x) = 5\sqrt{x-5} + 2$ is $[2,\infty)$.

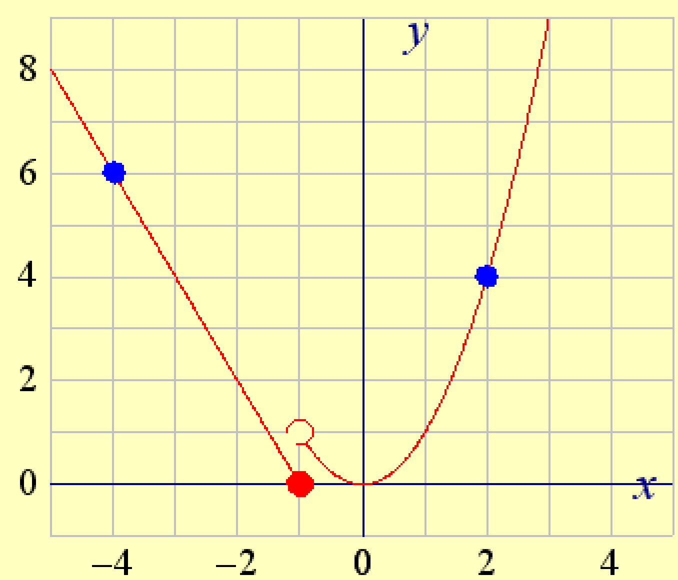
The graph of $g(x) = 5\sqrt{x-5} + 2$ is shown to the right, where the domain of g(x) is $[5,\infty)$ and the range of fg(x) is $[2,\infty)$.

Piecewise-Defined Functions

Sketch the graph of the function $f(x) = \begin{cases} -2x-2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Solution:

The function f is a linear function on the interval $(-\infty, -1]$ and a quadratic function on the interval $(-1, \infty)$. To graph f, we simply graph each portion separately, making sure that each formula is applied only on the appropriate interval. The complete graph appears on the next page, with the points $f(-4) = 6$ and $f(2) = 4$ noted in particular. Also note the use of a closed circle at $(-1, 0)$ to emphasize that this point is part of the graph, and the use of an open circle at $(-1, 1)$ to indicate that this point is *not* part of the graph. That is, the value of $f(-1)$ is 0, not 1.



$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ 2x + 6 & \text{if } x > 0 \end{cases}$$

Find: (a) $f(-5)$ (b) $f(0)$ (c) $f(5)$

(a) To find $f(-5)$, observe that when $x = -5$ the equation for f is given by $f(x) = x^2$. Therefore, we have the following.

$$f(-5) = (-5)^2 = 25$$

(b) When $x = 0$, the equation for f is $f(x) = -2$. Therefore, we have the following.

$$f(0) = -2$$

(c) When $x = 5$, the equation for f is $f(x) = 2x + 6$. Therefore, we have the following.

$$f(5) = 2(5) + 6 = 16$$

THE DIFFERENCE QUOTIENT

Find the difference quotient of f; that is, find $\frac{f(x+h) - f(x)}{h}$, $h \neq 0$, for the following function.

$$f(x) = 5x + 8$$

First, find the values of $f(x+h)$ and $f(x)$ so they can be substituted into the difference quotient $\frac{f(x+h) - f(x)}{h}$.

$$\begin{aligned} f(x+h) &= 5(x+h) + 8 \\ &= 5x + 5h + 8 \end{aligned}$$

So, $f(x+h) = 5x + 5h + 8$, and it is given that $f(x) = 5x + 8$.

Now, substitute $f(x+h) = 5x + 5h + 8$ and $f(x) = 5x + 8$ into the difference quotient $\frac{f(x+h) - f(x)}{h}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{(5x + 5h + 8) - (5x + 8)}{h}$$ Make the substitutions.

$$= \frac{5x + 5h + 8 - 5x - 8}{h}$$ Apply the distributive property to eliminate parentheses.

$$= \frac{5h}{h}$$ Combine like terms in the numerator.

$$= 5$$ Cancel out the common factor h.

Therefore, $\frac{f(x+h) - f(x)}{h} = 5$ when $f(x) = 5x + 8$.

If $f(x) = \frac{3x+2}{x-A}$ and $f(2) = -2$, what is the value of A?

An expression of the form f(x) means the value of the function f at the number x in its domain. The variable x is called the argument of the function.

The argument of the function $f(x) = \frac{3x+2}{x-A}$ is x

The value of f(2) can be found by substituting 2 for every occurrence of x in the function definition.

$$f(x) = \frac{3x+2}{x-A}$$

$$f(2) = \frac{3(2)+2}{(2)-A}$$

Notice that the value of f(2) is also given. Therefore, write an equation by setting the two expressions for f(2) equal to each other. Solve this equation for A.

$$-2 = \frac{3(2)+2}{(2)-A}$$

First, simplify the numerator and denominator on the right side. Evaluate the expression in the numerator.

$$-2 = \frac{3(2)+2}{(2)-A}$$

$$-2 = \frac{8}{2-A}$$

Multiply both sides by $2-A$ to clear the fraction and simplify.

$$(2-A)(-2) = \frac{8}{2-A}(2-A)$$

$$-4 + 2A = 8$$

To solve for A, add 4 to both sides, then divide both sides by 2.

$$-4 + 2A = 8$$

$$A = 6$$

Therefore, the value of A is 6.

Combining Functions

Addition, Subtraction, Multiplication, and Division of Functions

Let f and g be two functions. The **sum** $(f+g)$, **difference** $(f-g)$, **product** (fg), and **quotient** $\left(\frac{f}{g}\right)$ are four new functions defined as follows:

1. $(f+g)(x)=f(x)+g(x)$.
2. $(f-g)(x)=f(x)-g(x)$.
3. $(fg)(x)=f(x)\,g(x)$.
4. $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$, provided that $g(x)\neq 0$.

The domain of each of these new functions consists of the common elements of the domains of f and g individually, with the added condition that in the quotient function we have to omit those elements for which $g(x)=0$.

Given that $f(6)=-10$ and $g(6)=9$, find $(f-g)(6)$ and $\left(\frac{f}{g}\right)(6)$.

Solution:

By the definition of the difference and quotient of functions,

$$(f-g)(6)=f(6)-g(6)$$
$$=-10-(9)$$
$$=-19$$

and

$$\left(\frac{f}{g}\right)(6)=\left(\frac{f(6)}{g(6)}\right)$$
$$=\frac{-10}{9}.$$

Given the two functions $f(x) = \frac{1}{x}$ and $g(x) = [\![x + 3]\!]$, find $(f + g)(x)$ and $(fg)(x)$.

Solution:

By the definition of the sum and product of functions,

$$(f+g)(x) = f(x) + g(x) = \frac{1}{x} + [\![x+3]\!]$$

and

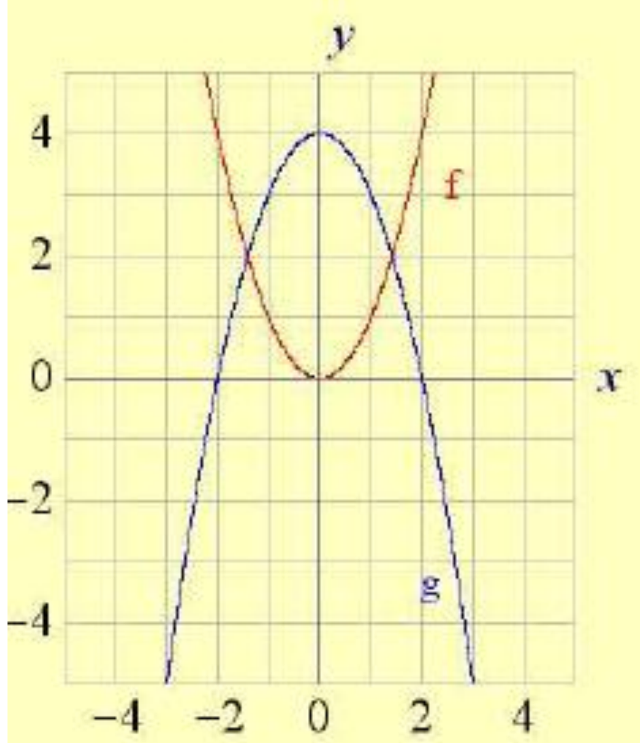
$$(fg)(x) = f(x) \cdot g(x) = \frac{1}{x} \cdot [\![x+3]\!] = \frac{[\![x+3]\!]}{x}.$$

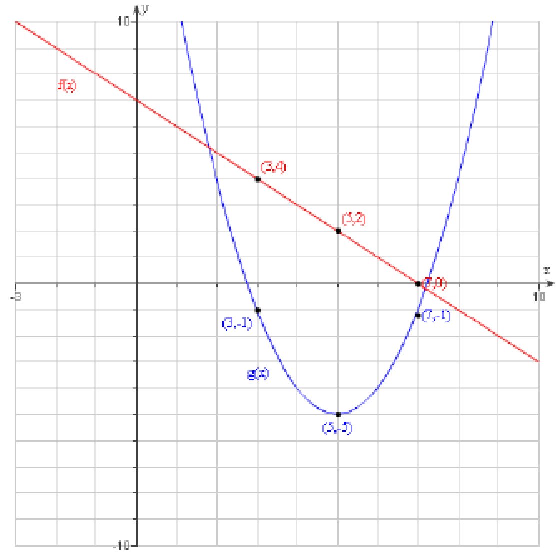
evaluate $\left(\frac{f}{g}\right)(-1)$.

it appears that $f(-1) = 1$ and $g(-1) = 3$, so

f(-1) =1 thus (-1,**1**) hence g(-1) = 3 thus (-1,**3**)

$$\left(\frac{f}{g}\right)(-1) = \frac{1}{3}.$$





(a) $(f+g)(3)$

To find $(f+g)(3)$, use the fact that $(f+g)(x) = f(x) + g(x)$. First, determine $f(3)$ and $g(3)$ from the graph.

$f(3) = 4$

$g(3) = -1$

Next, substitute the values into the equation $(f+g)(x) = f(x) + g(x)$ and simplify.

$$\begin{aligned}(f+g)(3) &= f(3) + g(3)\\ &= 4 + (-1)\\ &= 3\end{aligned}$$

(b) $(f+g)(5)$

To find $(f+g)(5)$, use the fact that $(f+g)(x) = f(x) + g(x)$. First, determine $f(5)$ and $g(5)$ from the graph.

$f(5) = 2$

$g(5) = -5$

Next, substitute the values into the equation $(f+g)(x) = f(x) + g(x)$ and simplify.

$$\begin{aligned}(f+g)(5) &= f(5) + g(5)\\ &= 2 + (-5)\\ &= -3\end{aligned}$$

(c) $(f-g)(7)$

To find $(f-g)(7)$, use the fact that $(f-g)(x) = f(x) - g(x)$. First, determine $f(7)$ and $g(7)$ from the graph.

$f(7) = 0$

$g(7) = -1$

Next, substitute the values into the equation $(f-g)(x) = f(x) - g(x)$ and simplify.

$$\begin{aligned}(f-g)(7) &= f(7) - g(7)\\ &= 0 - (-1)\\ &= 1\end{aligned}$$

(d) $(g-f)(7)$

To find $(g-f)(7)$, use the fact that $(g-f)(x) = g(x) - f(x)$. We determined that $f(7) = 0$ and $g(7) = -1$ in part c above.

Next, substitute the values into the equation $(f-g)(x) = f(x) - g(x)$ and simplify.

$$\begin{aligned}(g-f)(7) &= g(7) - f(7)\\ &= -1 - (0)\\ &= -1\end{aligned}$$

(e) $(f \cdot g)(3)$

To find $(f \cdot g)(3)$, use the fact that $(f \cdot g)(x) = f(x) \cdot g(x)$. We determined that $f(3) = 4$ and $g(3) = -1$ in part a above.

Next, substitute the values into the equation $(f \cdot g)(x) = f(x) \cdot g(x)$ and simplify.

$$\begin{aligned}(f \cdot g)(3) &= f(3) \cdot g(3)\\ &= (4)(-1)\\ &= -4\end{aligned}$$

(f) $\left(\frac{f}{g}\right)(5)$

To find $\left(\frac{f}{g}\right)(5)$, use the fact that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. We determined that $f(5) = 2$ and $g(5) = -5$ in part b above.

Next, substitute the values into the equation $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ and simplify.

$$\begin{aligned}\left(\frac{f}{g}\right)(5) &= \frac{f(5)}{g(5)}\\ &= \frac{2}{-5}\\ &= -\frac{2}{5}\end{aligned}$$

Composition of Functions

Let f and g be two functions. The **composition** of f and g, denoted $f \circ g$, is the function defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ consists of all x in the domain of g for which $g(x)$ is in turn in the domain of f. The function $f \circ g$ is read "f composed with g", or "f of g."

Another way of combining functions beside multiplication, division, addition and subtraction is in the form of COMPOSITION.
COMPOSITION of one function with another means apply one function **f(x)** to the output of another function **g(x).**
In COMPOSITION functions f(x) and g(x) are not commutative [a+b = b+a or ab= ba].

The diagram in Figure 1 is a sort of schematic of the composition of two functions. The ovals represent sets, with the leftmost oval being the domain of the function g. The arrows indicate the element that x is associated with by the various functions.

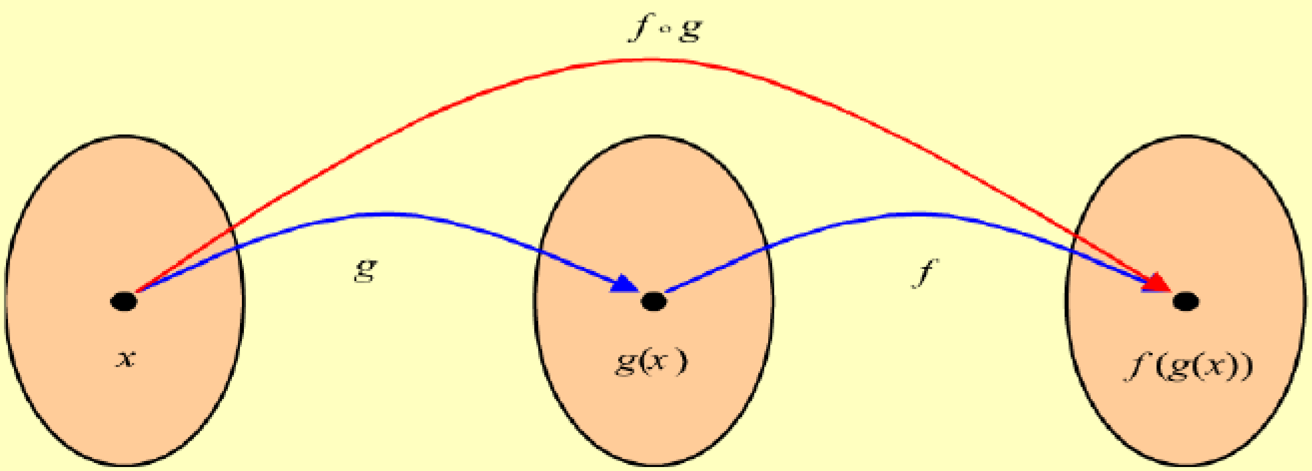


Figure 1: Composition of f and g

As with the four arithmetic ways of combining functions, we can evaluate the composition of two functions at a single point, or find a formula for the composition if we have been given formulas for the individual functions.

Given $f(x) = 2x^2 - 5$ and $g(x) = [\![x+2]\!]$, find:	**a.** $(f \circ g)(4)$ f of g
a. $(f \circ g)(4)$ x=4 then g(x) = x +2 → **4**+2	f(g(x)) then f(4+2)→ f(6) f(6) = 2x^2-5 → f(6) = 2(6)^2—5 → **67**

a. $g(4) = [\![4+2]\!] = 6$ and so $(f \circ g)(4) = f(g(4)) = f(6) = 67.$

a. $f(x) = x^2$ and $g(x) = 5x - 8$

$f(g(x)) =$ thus f(g(x)) = f(5x-8) hence we takes this value (5x-8)^2 or (5x-8)(5x-8) = **25x^2-80x+64**

Let $f(x) = |x - 2|$ and $g(x) = \frac{x+1}{3}$. Find formulas and state the domains for:

<table>
<tr>
<td>

a. $f \circ g$

Solution:

a. $$(f \circ g)(x) = f\left(\frac{x+1}{3}\right) = \left|\left(\frac{x+1}{3}\right) - 2\right| = \left|\frac{x-5}{3}\right|$$

g(x)= [(x+1)/ (3)] → f(g(x)) = f([(x+1)/ (3)])

f([(x+1)/ (3)]) = |x-2| → |[(x+1)/ (3)]) -2| →

|[(x+1)/ (3)]) -2|→ |[(x+1)/ (3)]) –(2/1) (3/3) |

|[(x+1)/ (3)]) –(6/3)|

|[(x-5)/ (3)]|

Always keep the absolute value sign.

</td>
<td>

b. $g \circ f$

b. $$(g \circ f)(x) = g(|x-2|) = \frac{|x-2|+1}{3}$$

f(x) = |x-2| → g(f(x)) = g(|x-2|)

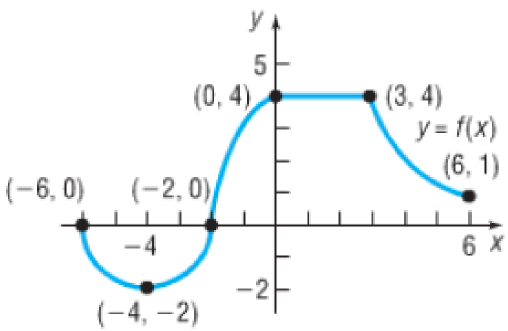
g(|x-2|) = [(x+1)/ (3)] → **[([|x-2|]+1)/ (3)]**

</td>
</tr>
</table>

3 Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant

Consider the graph given in Figure 18. If you look from left to right along the graph of the function, you will notice that parts of the graph are going up, parts are going down, and parts are horizontal. In such cases, the function is described as *increasing*, *decreasing*, or *constant*, respectively.

Figure 18



'LE 3 **Determining Where a Function Is Increasing, Decreasing, or Constant from Its Graph**

Where is the function in Figure 18 increasing? Where is it decreasing? Where is it constant?

Solution To answer the question of where a function is increasing, where it is decreasing, and where it is constant, we use strict inequalities involving the independent variable x, or we use open intervals* of x-coordinates. The function whose graph is given in Figure 18 is increasing on the open interval $(-4, 0)$ or for $-4 < x < 0$. The function is decreasing on the open intervals $(-6, -4)$ and $(3, 6)$ or for $-6 < x < -4$ and $3 < x < 6$. The function is constant on the open interval $(0, 3)$ or for $0 < x < 3$.

WARNING We describe the behavior of a graph in terms of its x-values. Do not say the graph in Figure 18 is increasing from the point $(-4, 2)$ to the point $(0, 4)$. Rather, say it is increasing on the interval $(-4, 0)$.

More precise definitions follow:

DEFINITIONS

A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function f is **constant** on an open interval I if, for all choices of x in I, the values $f(x)$ are equal.

Decreasing open intervals	Increasing open intervals	Constant open intervals
(-6,-4) = (-6, 0) → (-4,0) = -6 < x < -4 (3, 6) = (3, 0) → (6, 0) = 3 < x < 6	(-4, 0) = (-4, 0) → (0, 0) = -4 < x < 0	(0, 3) = (0, 0) → (3, 0) = 0 < x < 3

Inverses of Functions

Inverse of a Relation

Let R be a relation. The **inverse of** $\boldsymbol{R}$, denoted R^{-1}, is the set

$$R^{-1} = \{ (b, a) \mid (a, b) \in R \}.$$

Caution!

We are faced with another example of reuse of notation. f^{-1} does *not* stand for $\frac{1}{f}$! We use an exponent of -1 to indicate the reciprocal of a number or an algebraic expression, but when applied to a function or a relation it stands for the inverse relation.

The Horizontal Line Test

Let f be a function. We say that the graph of f passes the **horizontal line test** if every horizontal line in the plane intersects the graph no more than once.

Finding Inverse Functions

Let f be a one-to-one function, and assume that f is defined by a formula. To find a formula for f^{-1}, perform the following steps:

1. Replace $f(x)$ in the definition of f with the variable y. The result is an equation in x and y that is solved for y at this point.
2. Interchange x and y in the equation.
3. Solve the new equation for y.
4. Replace the y in the remaining equation with $f^{-1}(x)$.

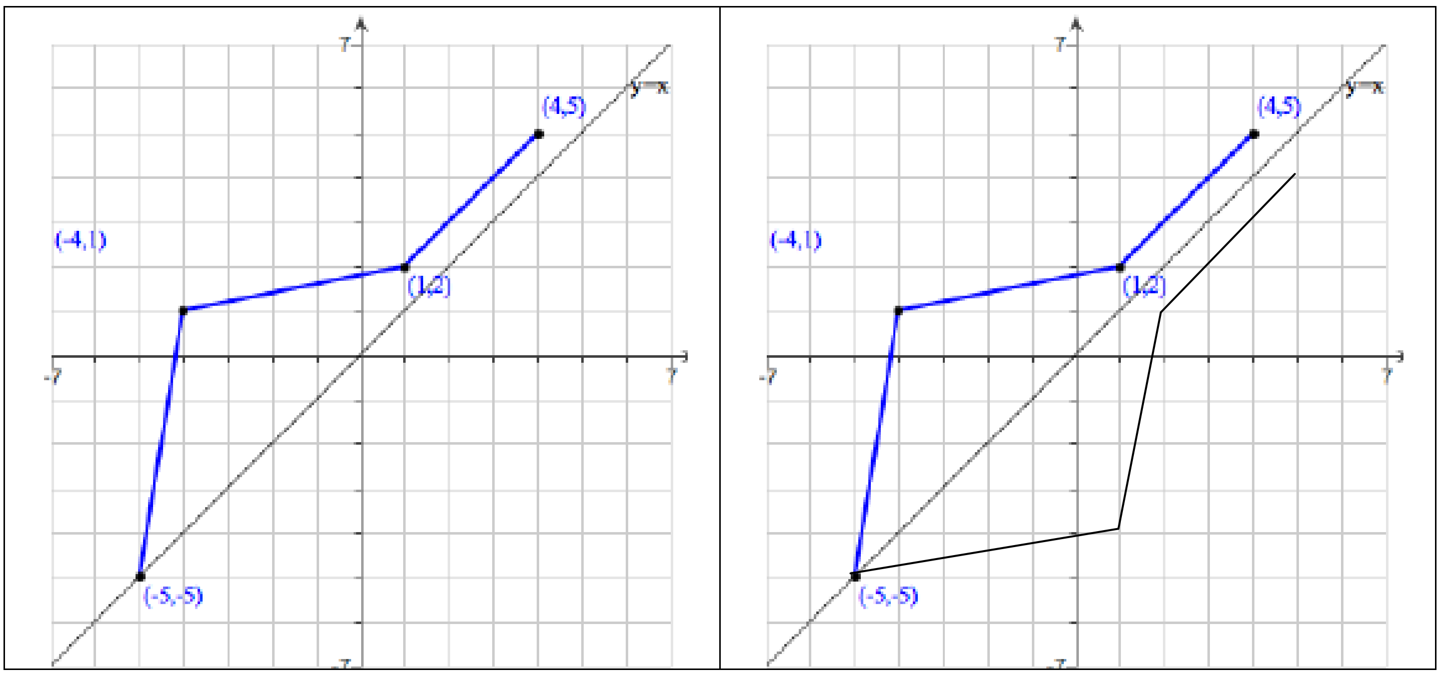
FIND THE INVERSE OF:

b.

$h(x) = \dfrac{x-3}{2}$	We will use the algorithm for this function.
$y = \dfrac{x-3}{2}$	The first step is to replace $h(x)$ with y.
$x = \dfrac{y-3}{2}$	The second step is to interchange x and y in the equation.
$2x = y - 3$	Cross multiply
$y = 2x + 3$	We now have to solve the equation for y.
$h^{-1}(x) = 2x + 3$	The last step is to name the formula h^{-1}.

Use the given graph of $y = f(x)$ to evaluate the following and graph the inverse of $f(x)$.

(a) $f(-5)$ (b) $f(3)$ (c) $f^{-1}(1)$ (d) $f^{-1}(-5)$



Y = f(x)

X	Y
4	5
1	2
-4	1
-5	-5

Y = $f^{-1}(x)$

X	Y
5	4
2	1
1	-4
-5	-5

(a) f (-5) point → (-5, 5)

(b) f (3) point → (3, 4)

(c) $f^{-1}(1)$ point → (1, -4)

(d) $f^{-1}(-5)$ point → (-5, -5)