CHAPTER 1~ GRAPHS AND CIRCLES

The **standard form of an equation of a circle** with radius r and center (h,k) is

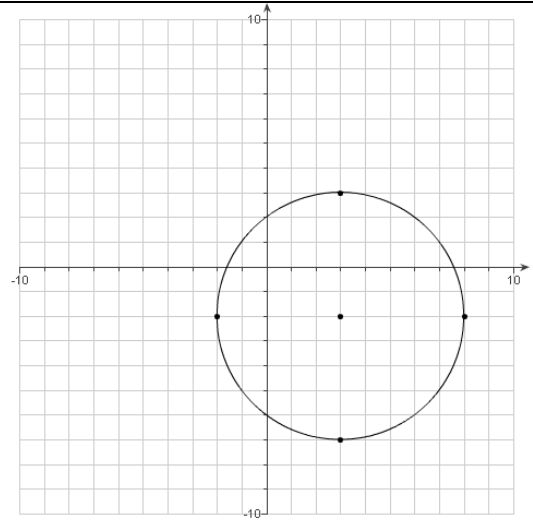
$(x-h)^2+(y-k)^2=r^2$

When its graph is a circle, the equation

$x^2+y^2+ax+by+c=0$

is referred to as the **general form of the equation of a circle.**

A circle has the equation $6x^2+6y^2-36x+24y-72=0$. Graph the circle using the center (h,k) and radius r. Find the intercepts, if any, of the graph.



To find the center (h,k) and radius r of the circle, compare the given equation to the standard form of the equation of a circle.

First, divide both sides by the greatest common factor of each term, 6.

$x^2+y^2-6x+4y-12=0$

Next, complete the square in both x and y to put the equation in standard form. Group the expressions involving x, group the expressions involving y, and put the constant on the right side of the equation.

$(x^2-6x)+(y^2+4y)=12$

Then complete the square of each expression in parentheses on the left side. Remember that any number added on the left side of the equation must be added on the right.

$(x^2-6x+9)+(y^2+4y+4)=12+9+4$

Now factor to put the equation in standard form.

$$(x^2-6x+9)+(y^2+4y+4) = 12+9+4$$
$$(x-3)^2+(y+2)^2 = 25$$
$$(x-h)^2+(y-k)^2 = r^2$$

The center is $(h,k)=(3,-2)$ and the radius $r=5$.

To graph the circle, first plot the center $(3,-2)$. Since the radius is 5, we can then locate four points on the circle by plotting points 5 units to the left, to the right, up, and down from the center.

The graph of the circle is shown on the right.

To find the x-intercepts, if any, let $y=0$ in the standard form of the equation.

$(x-3)^2+(0+2)^2 = 25$ Let $y=0$.

$(x-3)^2+4 = 25$ Simplify.

$(x-3)^2 = 21$ Subtract 4 from both sides.

Solve for x.

$(x-3)^2 = 21$

$x-3 = \pm\sqrt{21}$ Apply the square root method.

$x = 3\pm\sqrt{21}$ Add 3 to both sides.

The x-intercepts are $x=3-\sqrt{21}$ and $3+\sqrt{21}$.

Therefore, the x-intercepts occur at the following points.

$(3-\sqrt{21},0)$, $(3+\sqrt{21},0)$

To find the y-intercepts, if any, let $x=0$ in the standard form of the equation.

$(0-3)^2+(y+2)^2 = 25$ Let $x=0$.

$9+(y+2)^2 = 25$ Simplify.

$(y+2)^2 = 16$ Subtract 9 from both sides.

Solve for y.

$(y+2)^2 = 16$

$y+2 = \pm 4$ Apply the square root method.

$y = -2\pm 4$ Subtract 2 from both sides.

The y-intercepts are $y=-6$ and 2.

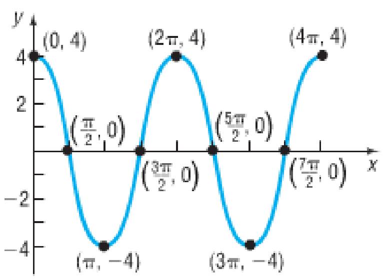
The y-intercepts occur at the following points.

$(0,-6)$, $(0,2)$

about a function if its graph is given.

EXAMPLE 2 Obtaining Information from the Graph of a Function

Figure 15



Let f be the function whose graph is given in Figure 15. (The graph of f might represent the distance y that the bob of a pendulum is from its *at-rest* position at time x. Negative values of y mean that the pendulum is to the left of the at-rest position, and positive values of y mean that the pendulum is to the right of the at-rest position.)

(a) What are $f(0)$, $f\left(\frac{3\pi}{2}\right)$, and $f(3\pi)$?

(b) What is the domain of f?

(c) What is the range of f?

(d) List the intercepts. (Recall that these are the points, if any, where the graph crosses or touches the coordinate axes.)

(e) How many times does the line $y = 2$ intersect the graph?

(f) For what values of x does $f(x) = -4$?

(g) For what values of x is $f(x) > 0$?

Solution

(a) Since $(0, 4)$ is on the graph of f, the y-coordinate 4 is the value of f at the x-coordinate 0; that is, $f(0) = 4$. In a similar way, we find that when $x = \frac{3\pi}{2}$, then $y = 0$, so $f\left(\frac{3\pi}{2}\right) = 0$. When $x = 3\pi$, then $y = -4$, so $f(3\pi) = -4$.

(b) To determine the domain of f, we notice that the points on the graph of f have x-coordinates between 0 and 4π, inclusive; and for each number x between 0 and 4π, there is a point $(x, f(x))$ on the graph. The domain of f is $\{x \mid 0 \le x \le 4\pi\}$ or the interval $[0, 4\pi]$.

(c) The points on the graph all have y-coordinates between -4 and 4, inclusive; and for each such number y, there is at least one number x in the domain. The range of f is $\{y \mid -4 \le y \le 4\}$ or the interval $[-4, 4]$.

(d) The intercepts are the points

$$(0, 4), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{5\pi}{2}, 0\right), \quad \text{and} \quad \left(\frac{7\pi}{2}, 0\right)$$

(e) If we draw the horizontal line $y = 2$ on the graph in Figure 15, we find that it intersects the graph four times.

(f) Since $(\pi, -4)$ and $(3\pi, -4)$ are the only points on the graph for which $y = f(x) = -4$, we have $f(x) = -4$ when $x = \pi$ and $x = 3\pi$.

(g) To determine where $f(x) > 0$, look at Figure 15 and determine the x-values from 0 to 4π for which the y-coordinate is positive. This occurs on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \left(\frac{7\pi}{2}, 4\pi\right]$. Using inequality notation, $f(x) > 0$ for $0 \le x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < \frac{5\pi}{2}$ or $\frac{7\pi}{2} < x \le 4\pi$.

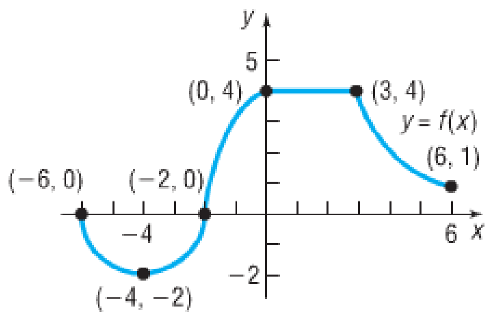
When the graph of a function is given, its domain may be viewed as the shadow created by the graph on the x-axis by vertical beams of light. Its range can be viewed as the shadow created by the graph on the y-axis by horizontal beams of light. Try this technique with the graph given in Figure 15.

Now Work PROBLEMS 9 AND 13

3 Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant

Consider the graph given in Figure 18. If you look from left to right along the graph of the function, you will notice that parts of the graph are going up, parts are going down, and parts are horizontal. In such cases, the function is described as *increasing, decreasing,* or *constant,* respectively.

Figure 18



LE 3 Determining Where a Function Is Increasing, Decreasing, or Constant from Its Graph

Where is the function in Figure 18 increasing? Where is it decreasing? Where is it constant?

Solution

WARNING We describe the behavior of a graph in terms of its x-values. Do not say the graph in Figure 18 is increasing from the point $(-4, 2)$ to the point $(0, 4)$. Rather, say it is increasing on the interval $(-4, 0)$. ■

To answer the question of where a function is increasing, where it is decreasing, and where it is constant, we use strict inequalities involving the independent variable x, or we use open intervals* of x-coordinates. The function whose graph is given in Figure 18 is increasing on the open interval $(-4, 0)$ or for $-4 < x < 0$. The function is decreasing on the open intervals $(-6, -4)$ and $(3, 6)$ or for $-6 < x < -4$ and $3 < x < 6$. The function is constant on the open interval $(0, 3)$ or for $0 < x < 3$.

More precise definitions follow:

DEFINITIONS

A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function f is **constant** on an open interval I if, for all choices of x in I, the values $f(x)$ are equal.

Increasing	Decreasing	Constant
(**-4**,-2)→ (**0**,4) -4 < x < 0	(**-6**,0)→(**-4**,-2) & (**3**,4) → (**6**,1) -6<x<-4 & 3<x<6	(**0**,4)→(**3**,4) 0<x<3

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ 2x + 6 & \text{if } x > 0 \end{cases}$$

Find: (a) $f(-5)$ (b) $f(0)$ (c) $f(5)$

(a) To find $f(-5)$, observe that when $x = -5$ the equation for f is given by $f(x) = x^2$. Therefore, we have the following.

$$f(-5) = (-5)^2 = 25$$

(b) When $x = 0$, the equation for f is $f(x) = -2$. Therefore, we have the following.

$$f(0) = -2$$

(c) When $x = 5$, the equation for f is $f(x) = 2x + 6$. Therefore, we have the following.

$$f(5) = 2(5) + 6 = 16$$

Find the difference quotient of f; that is, find $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$, for the following function.

$f(x) = 5x+8$

First, find the values of $f(x+h)$ and $f(x)$ so they can be substituted into the difference quotient $\frac{f(x+h)-f(x)}{h}$.

$$\begin{aligned} f(x+h) &= 5(x+h)+8 \\ &= 5x+5h+8 \end{aligned}$$

So, $f(x+h) = 5x+5h+8$, and it is given that $f(x) = 5x+8$.

Now, substitute $f(x+h) = 5x+5h+8$ and $f(x) = 5x+8$ into the difference quotient $\frac{f(x+h)-f(x)}{h}$.

$\frac{f(x+h)-f(x)}{h}$	$= \frac{(5x+5h+8)-(5x+8)}{h}$	Make the substitutions.
	$= \frac{5x+5h+8-5x-8}{h}$	Apply the distributive property to eliminate parentheses.
	$= \frac{5h}{h}$	Combine like terms in the numerator.
	$= 5$	Cancel out the common factor h.

Therefore, $\frac{f(x+h)-f(x)}{h} = 5$ when $f(x) = 5x+8$.

If $f(x) = \frac{3x+2}{x-A}$ and $f(2) = -2$, what is the value of A?

An expression of the form f(x) means the value of the function f at the number x in its domain. The variable x is called the argument of the function.

The argument of the function $f(x) = \frac{3x+2}{x-A}$ is x.

The value of f(2) can be found by substituting 2 for every occurrence of x in the function definition.

$$f(x) = \frac{3x+2}{x-A}$$
$$f(2) = \frac{3(2)+2}{(2)-A}$$

Notice that the value of f(2) is also given. Therefore, write an equation by setting the two expressions for f(2) equal to each other. Solve this equation for A.

$$-2 = \frac{3(2)+2}{(2)-A}$$

First, simplify the numerator and denominator on the right side. Evaluate the expression in the numerator.

$$-2 = \frac{3(2)+2}{(2)-A}$$
$$-2 = \frac{8}{2-A}$$

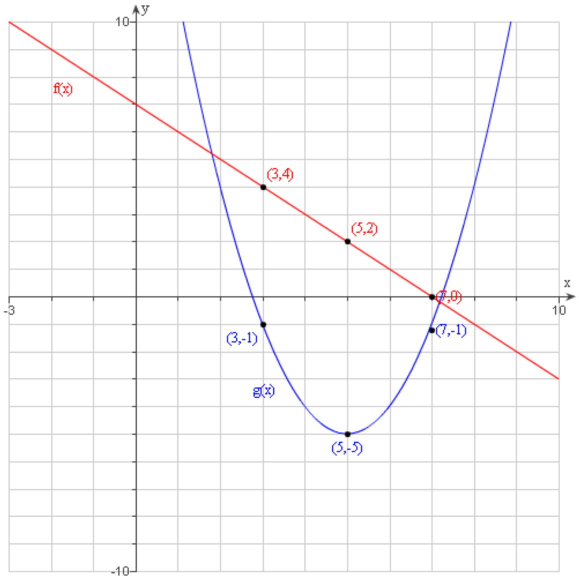
Multiply both sides by 2 − A to clear the fraction and simplify.

$$(2-A)(-2) = \frac{8}{2-A}(2-A)$$
$$-4 + 2A = 8$$

To solve for A, add 4 to both sides, then divide both sides by 2.

$$-4 + 2A = 8$$
$$A = 6$$

Therefore, the value of A is 6.



(a) $(f+g)(3)$

To find $(f+g)(3)$, use the fact that $(f+g)(x)=f(x)+g(x)$. First, determine $f(3)$ and $g(3)$ from the graph.

$f(3)=4$

$g(3)=-1$

Next, substitute the values into the equation $(f+g)(x)=f(x)+g(x)$ and simplify.

$$\begin{aligned}(f+g)(3) &= f(3)+g(3)\\ &= 4+(-1)\\ &= 3\end{aligned}$$

(b) $(f+g)(5)$

To find $(f+g)(5)$, use the fact that $(f+g)(x)=f(x)+g(x)$. First, determine $f(5)$ and $g(5)$ from the graph.

$f(5)=2$

$g(5)=-5$

Next, substitute the values into the equation $(f+g)(x)=f(x)+g(x)$ and simplify.

$$\begin{aligned}(f+g)(5) &= f(5)+g(5)\\ &= 2+(-5)\\ &= -3\end{aligned}$$

(c) $(f-g)(7)$

To find $(f-g)(7)$, use the fact that $(f-g)(x)=f(x)-g(x)$. First, determine $f(7)$ and $g(7)$ from the graph.

$f(7)=0$

$g(7)=-1$

Next, substitute the values into the equation $(f-g)(x)=f(x)-g(x)$ and simplify.

$$\begin{aligned}(f-g)(7) &= f(7)-g(7)\\ &= 0-(-1)\\ &= 1\end{aligned}$$

(d) $(g-f)(7)$

To find $(g-f)(7)$, use the fact that $(g-f)(x)=g(x)-f(x)$. We determined that $f(7)=0$ and $g(7)=-1$ in part c above.

Next, substitute the values into the equation $(f-g)(x)=f(x)-g(x)$ and simplify.

$$\begin{aligned}(g-f)(7) &= g(7)-f(7)\\ &= -1-(0)\\ &= -1\end{aligned}$$

(e) $(f\cdot g)(3)$

To find $(f\cdot g)(3)$, use the fact that $(f\cdot g)(x)=f(x)\cdot g(x)$. We determined that $f(3)=4$ and $g(3)=-1$ in part a above.

Next, substitute the values into the equation $(f\cdot g)(x)=f(x)\cdot g(x)$ and simplify.

$$\begin{aligned}(f\cdot g)(3) &= f(3)\cdot g(3)\\ &= (4)(-1)\\ &= -4\end{aligned}$$

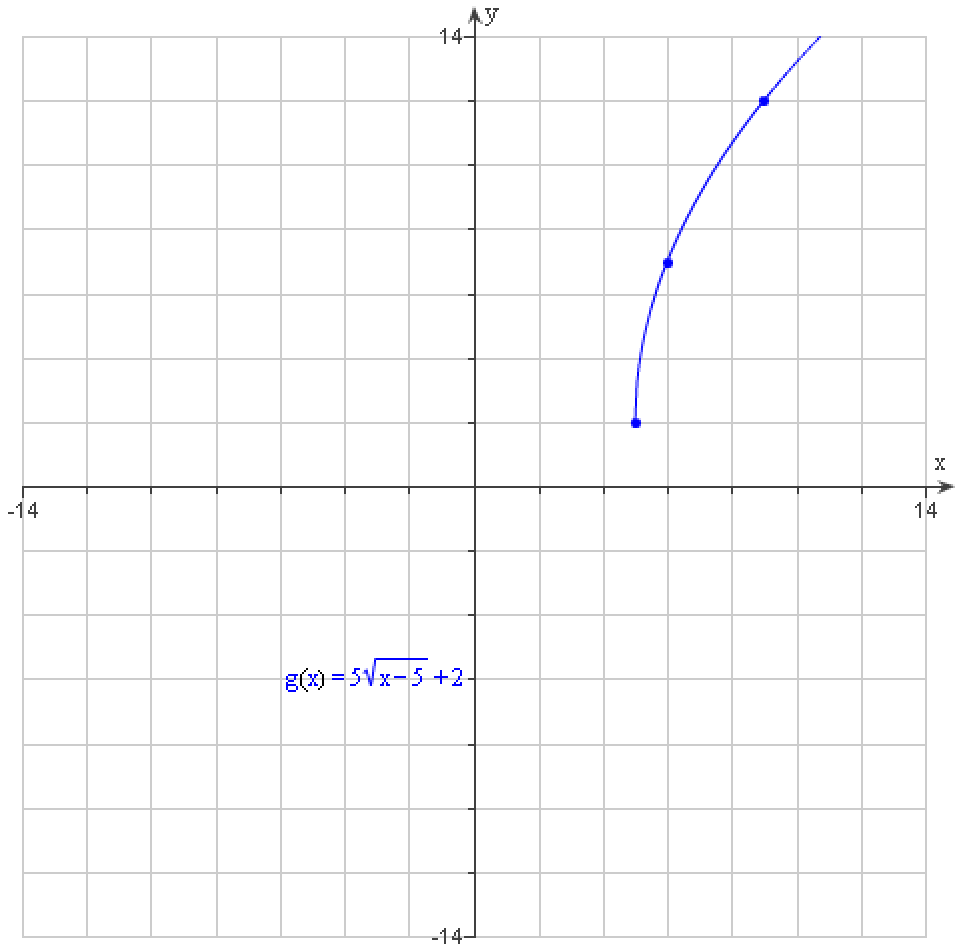
(f) $\left(\frac{f}{g}\right)(5)$

To find $\left(\frac{f}{g}\right)(5)$, use the fact that $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$. We determined that $f(5)=2$ and $g(5)=-5$ in part b above.

Next, substitute the values into the equation $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$ and simplify.

$$\begin{aligned}\left(\frac{f}{g}\right)(5) &= \frac{f(5)}{g(5)}\\ &= \frac{2}{-5}\\ &= -\frac{2}{5}\end{aligned}$$

Domain and Range. Graphs and Functions. 2.5



find the domain and range of g(x).

The domain of g is the largest set of real numbers for which the value of g(x) is a real number.

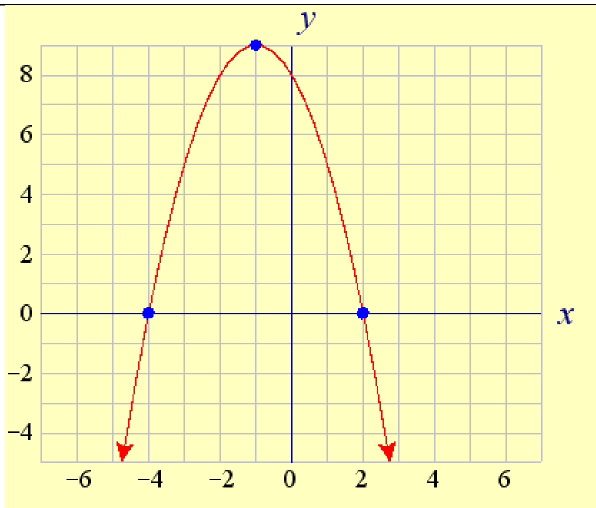
Remember that the square root function is undefined for numbers less than zero. The function $g(x) = 5\sqrt{x-5} + 2$ is defined for all real numbers greater than or equal to 5. Therefore, the domain of $g(x) = 5\sqrt{x-5} + 2$ is $[5,\infty)$.

The range of g is the set of g(x)-values of the function that are images of the x-values in the domain.

The smallest value of $g(x) = 5\sqrt{x-5} + 2$ occurs at the point (5,2) and the values of g(x) continue to infinity. Therefore, the range of $g(x) = 5\sqrt{x-5} + 2$ is $[2,\infty)$.

The graph of $g(x) = 5\sqrt{x-5} + 2$ is shown to the right, where the domain of g(x) is $[5,\infty)$ and the range of fg(x) is $[2,\infty)$.

Determine the quadratic function whose graph is given the vertex is (-1,9) and y- intercept is (0,8)



The basic function of this graph is -x^2,

Next the quadratic should look in the form of

f(x) = a (x-h)^2 +k

We know the vertex is (h,k) → (-1,9)

Horizontal movement

f(x) = a (x-**h**)^2 +k

When "**h**" is positive the equation moves to the left and when "**h**" is negative the equation moves to the right.

Since the graph shifted to the left and the value for "**x**" in the vertex is -1 then "h" is positive and the value for "h" is 1. [x-h=0 thus -1+1 = 0]

Vertical Movement

f(x) = a (x-h)^2 +**k**

When "**k**" is positive the function moves up, and when "**k**" is negative the function moves down.

Since the graph shifted up "**k**" is positive.

Reflection over the x-axis

f(x) = **a** (x-h)^2 + k

When ever "**a**" is negative thus "**-a**"the function is reflected over the x-axis
If "**a**" is omitted then just keep the negative sign.

The formula will change from f(x) = a (x-**h**)^2 **+k** to f(x) = **-a** (x**+1**)^2 **+9** → f(x) = **-**(x**+1**)^2 **+9**

To test for the y-intercept replace "x" with 0 → f(0) = **-**(0**+1**)^2 **+9** → -1 +9 = 8 thus **(0,8)**