2.1 Lines

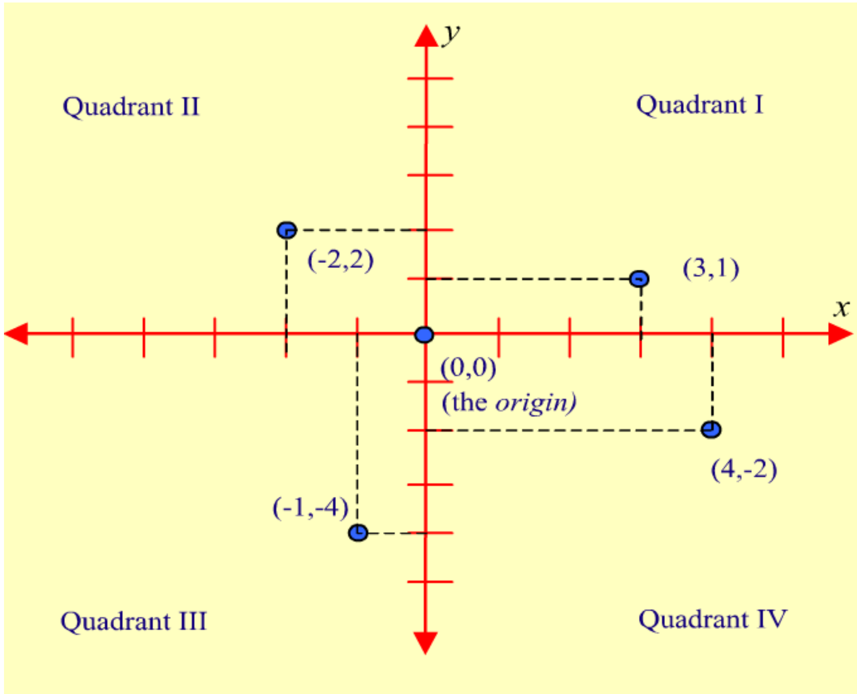
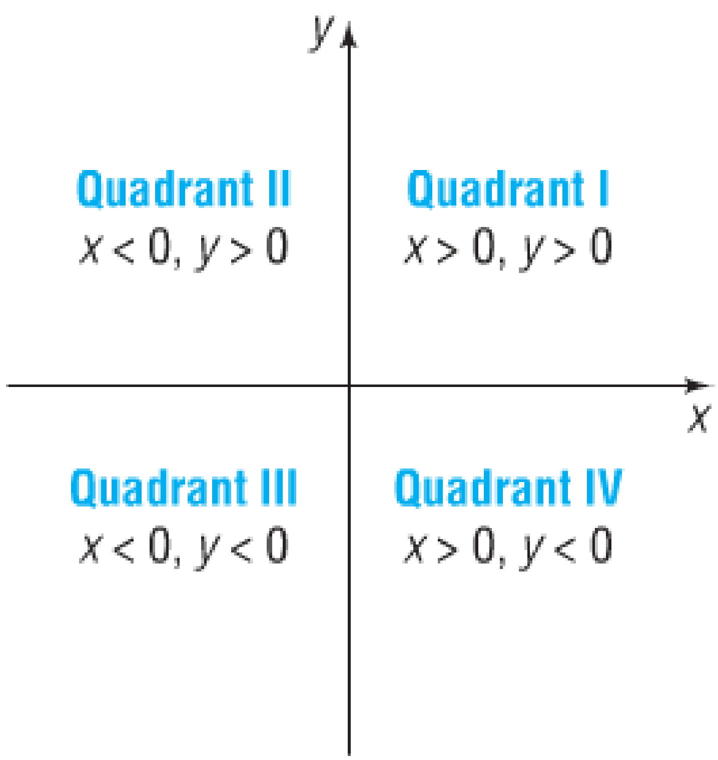
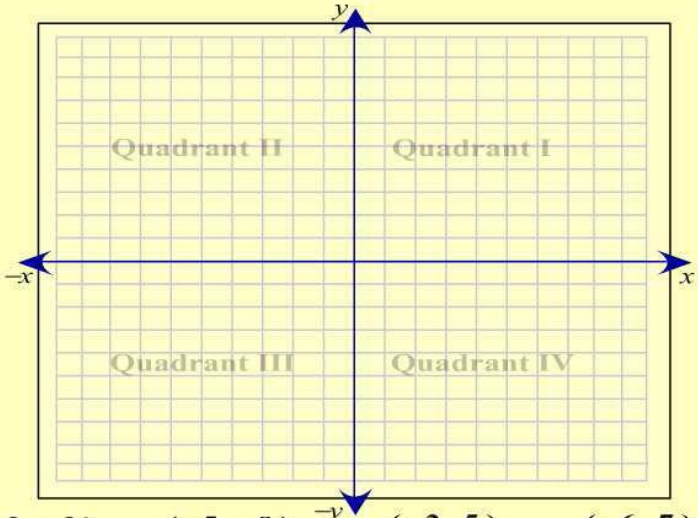


Figure 1: The Cartesian Plane $\mathbb{R}^2$



Now it's your turn to practice what you just learned.
Click on a point and drag it into the appropriate quadrant. You do not need to plot the point, just place it in the proper region of the Cartesian coordinate system.



(2 , −3) (−5 , −8) (−2 , 5) (−6 , 7)
(3 , 10) (8 , 4) (10 , −6) (−9 , −9)

The lengths of the two perpendicular sides of the triangle are easily calculated, as these lengths correspond to distances between numbers on real number lines. (The absolute value symbols are present as we don't necessarily know, for instance, whether $x_2 - x_1$ or $x_1 - x_2$ is non-negative.) Now, we can apply the Pythagorean Theorem to determine the distance labeled d in Figure 2:

$$d^2 = \left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2$$

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

(Notice that the absolute value symbols are not necessary in the formula, as any quantity squared is automatically non-negative.)

Distance Formula

Letting $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$ represent two points on the Cartesian plane, the distance between these two points may be found using the following formula, derived from the Pythagorean Theorem:

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

Throughout the rest of this course, we will have reasons for wanting to know, on occasion, the *distance* between two points in the Cartesian plane. We already have the tools necessary to answer this question, and we will now derive a formula that we can apply whenever necessary.

Let (x_1, y_1) and (x_2, y_2) be the coordinates of two arbitrary points in the plane. By drawing the dotted lines parallel to the coordinate axis as shown in Figure 2, we can form a right triangle. Note that we are able to determine the coordinates of the vertex at the right angle from the two vertices (x_1, y_1) and (x_2, y_2).

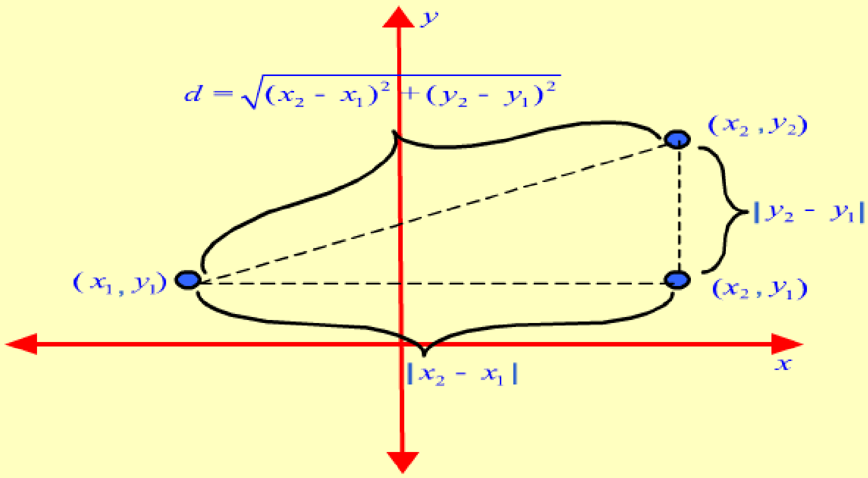


Figure 2: Distance Formula

Find the distance d between the points $(-4, 5)$ and $(3, 2)$.

$$d = \sqrt{[3 - (-4)]^2 + (2 - 5)^2} = \sqrt{7^2 + (-3)^2}$$
$$= \sqrt{49 + 9} = \sqrt{58} \approx 7.62$$

Since x_1 and x_2 are numbers on a real number line, and y_1 and y_2 are numbers on a (different) real number line, determining the averages of these two pairs of numbers is straightforward: the average of the x-coordinates is $\dfrac{x_1 + x_2}{2}$ and the average of the y-coordinates is $\dfrac{y_1 + y_2}{2}$. Putting these two coordinates together gives us the desired formula:

Midpoint Formula

Letting (x_1, y_1) and (x_2, y_2) represent two points on the Cartesian plane, the midpoint between these two points may be found using the following formula, which finds the average of the two x-values and the average of the two y-values:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

We will also, on occasion, want to be able to determine the *midpoint* of a line segment in the plane. That is, given two points (x_1, y_1) and (x_2, y_2), we will want to know the coordinates of the point exactly halfway between the two given points. Of course, such a formula will logically take the form of an ordered pair, as it must give us two numbers.

Consider the points as plotted in Figure 3. The x-coordinate of the midpoint should be the average of the two x-coordinates of the given points, and similarly for the y-coordinate.

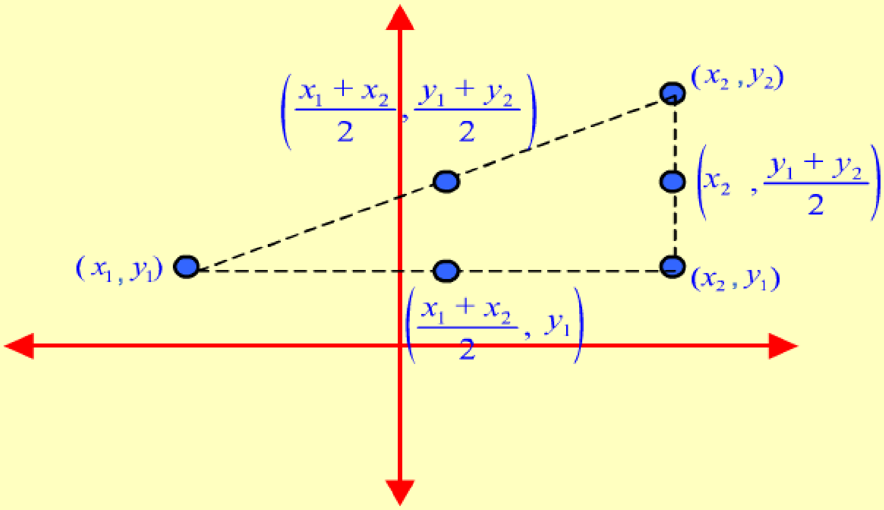
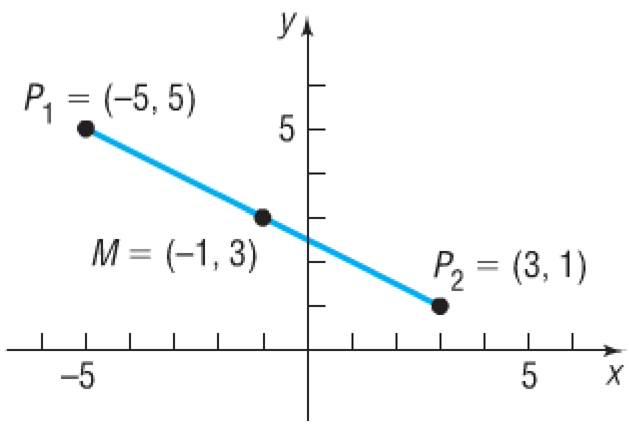


Figure 3: Midpoint Formula

Find the midpoint of the line segment from $P_1 = (-5, 5)$ to $P_2 = (3, 1)$. Plot the points P_1 and P_2 and their midpoint.

$$x = \frac{x_1 + x_2}{2} = \frac{-5 + 3}{2} = -1 \quad \text{and} \quad y = \frac{y_1 + y_2}{2} = \frac{5 + 1}{2} = 3$$



Determine: **a.** the distance between (2, 0) and (–3, 4).

b. the midpoint of the line segment joining (0, 2) and (–3, 4).

2.1

Linear Equations in Two Variables

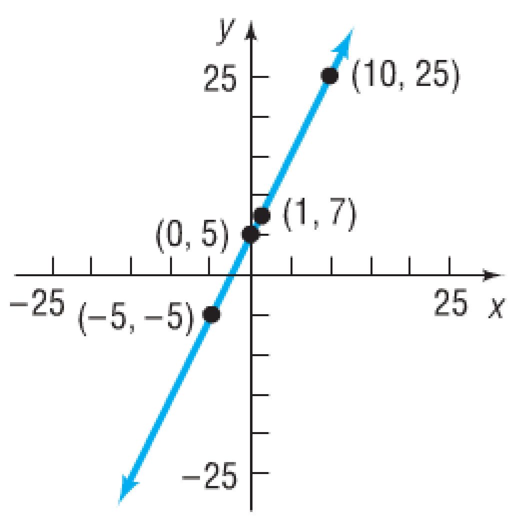
A **linear equation in two variables**, say the variables x and y, is an equation that can be written in the form

$$ax + by = c,$$

where a, b, and c are constants and a and b are not both zero. This form of such an equation is called the **standard form**.

Graph the equation: $y = 2x + 5$

If	Then	Point on Graph
$x = 0$	$y = 2(0) + 5 = 5$	$(0, 5)$
$x = 1$	$y = 2(1) + 5 = 7$	$(1, 7)$
$x = -5$	$y = 2(-5) + 5 = -5$	$(-5, -5)$
$x = 10$	$y = 2(10) + 5 = 25$	$(10, 25)$



Determine if the following equation is a linear equation.

a. $-(3-5x)+4y=2x+y-2$

Use algebra to rewrite the equation in an equivalent form which will be clearly either linear or not.

Determine if the following equation is a linear equation.

b. $x+2y-2(y-3)=4x$

Proceed as in the problem in the previous screen, using familiar properties to rewrite the equation in standard form.

Determine if the following equation is a linear equation.

c. $\frac{x}{3}+y=\frac{y-2}{2}$

Although this equation may appear at first to not fit the criteria to be linear, appearances are deceiving.

Determine if the following equations are linear equations.

d. $4x+(2x+5)-2y=-2y+3(2x-3)$

e. $3x^3+5y=6x$

Intercepts of the Coordinate Axes

Quite often, the goal in working with a given linear equation is to graph its solution set. If the straight line whose points constitute the solution set crosses the horizontal and vertical axes in two distinct points, knowing the coordinates of these two points is sufficient to graph the complete solution; this follows from the fact that two points are enough to completely determine a line.

If the equation under consideration is in the two variables x and y, it is natural to call the point where the graph crosses the x-axis the ***x*-intercept**, and the point where it crosses the y-axis the ***y*-intercept**. If the line does indeed cross both axes, the two intercepts are easy to find:

the y-coordinate of the x-intercept is 0, and the x-coordinate of the y-intercept is 0.

Find the x- and y- intercepts of the following equations, and then graph each equation.

a. $4x + 5y = 20$

Find the x- and y- intercepts of the following equation, and then graph.

b. $2x - (7 - x) - 3y = 2$

Horizontal and Vertical Lines

The above procedure for graphing linear equations may be undesirable or inadequate for several reasons. One such reason is that the x-intercept and y-intercept of a given line may be the same point: the origin. In order to graph the equation in this case, a second point distinct from the origin must be found in order to have two points to connect with a line. Another reason is that a given equation may not have one of the two types of intercepts. A moment's thought will reveal that this can happen only when the graph of the equation is a horizontal or vertical line.

Equations whose graphs are horizontal or vertical lines are missing one of the two variables (though it may be necessary to simplify the equation before this is apparent).

However, in the absence of any other information, it is impossible to know if the solution of such a linear equation consists of a point on the real number line or a line in the Cartesian plane (or, indeed, of a higher-dimensional set in a higher-dimensional coordinate system!). You must rely on the context of the problem to know how many variables should be considered. Throughout this chapter, all equations are assumed to be in two variables unless otherwise stated, so an equation of the form $ax = c$ or $by = c$ should be thought of as representing a line in $\mathbb{R}^2$.

Consider an equation of the form $ax = c$. The variable y is absent, so *any* value for this variable will suffice as long as we pair it with $x = \frac{c}{a}$. Thinking of the solution set as a set of ordered pairs, the solution consists of ordered pairs with a fixed first coordinate and *arbitrary* second coordinate.

This describes, geometrically, a vertical line with an x-intercept of $\frac{c}{a}$.

Similarly, the equation $by = c$ represents a horizontal line with y-intercept equal to $\frac{c}{b}$.

Graph the following equations.

a. $5y - 3(y - 4) + 4x = 2y$ **b.** $-1 + 2y = 6$ **c.** $3x = 0$

2.3 SLOPE OF A LINE

The Slope of a Line

Let L stand for a given line in the Cartesian plane, and let (x_1, y_1) and (x_2, y_2) be the coordinates of two distinct points on L. The **slope** of the line L is the ratio $\frac{y_2 - y_1}{x_2 - x_1}$, which can be described in words as "change in y over change in x" or "rise over run."

The phrase "rise over run" is motivated by the diagram in Figure 1.

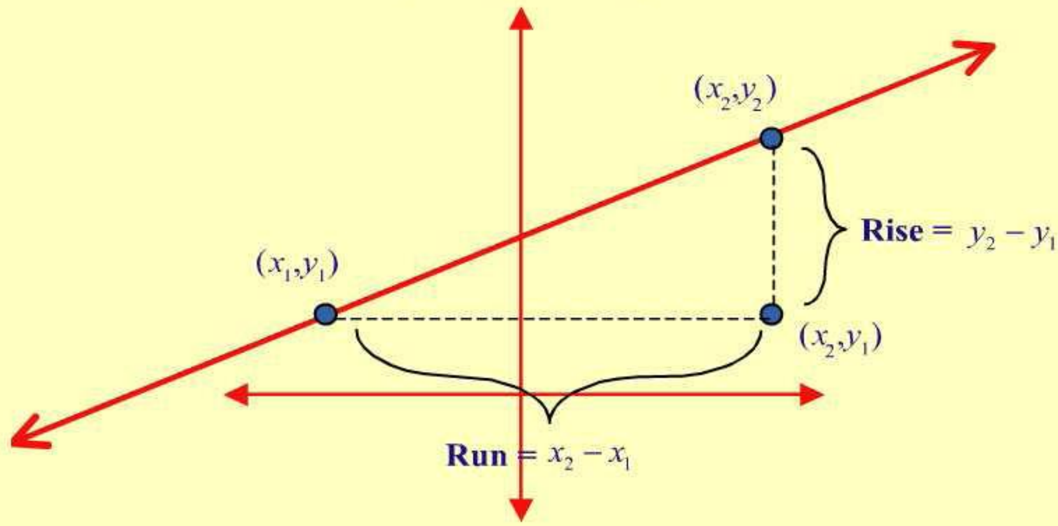
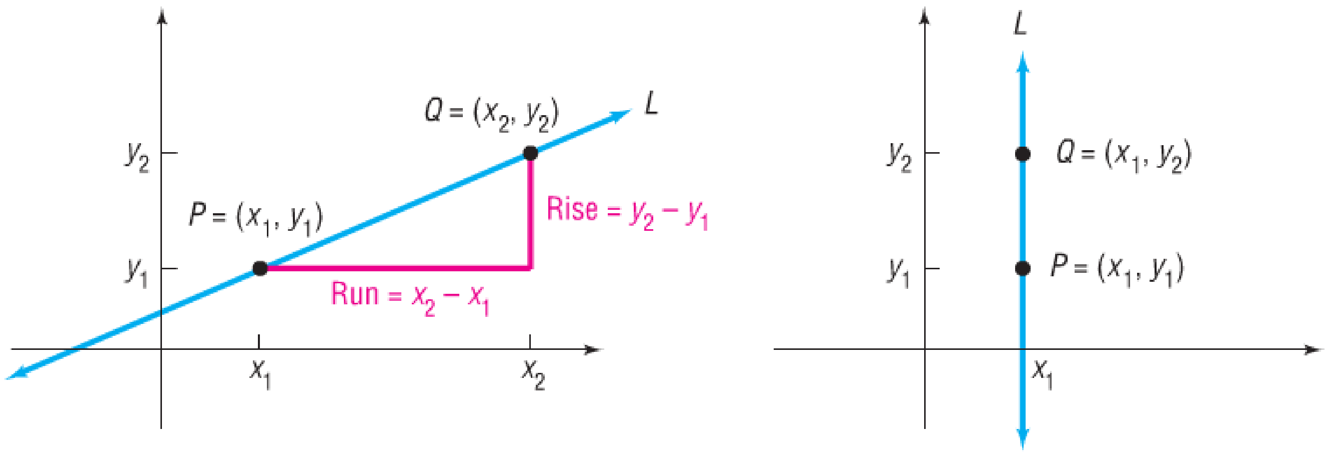


Figure 1: Rise and Run Between Two Points

As drawn in Figure 1, the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is positive, and we say that the line has positive slope. If the rise and run have opposite sign, the slope of the line would be negative and the line under consideration would be falling from the upper left to the lower right.



(a) Slope of L is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ **(b)** Slope is undefined; L is vertical

Determine the slopes of the lines passing through the following pairs of points in $\mathbb{R}^2$.

a. $(7, -1)$ and $(-3, -2)$ **b.** $\left(2, \frac{1}{3}\right)$ and $\left(\frac{5}{4}, -1\right)$

c. $(3, 6)$ and $(-4, 6)$

Determine the slopes of the lines defined by the following equations.

a. $5x - 2y = 10$ **b.** $3x + 4y = 7$ **c.** $x = \dfrac{5}{3}$

Slope intercept form of a line

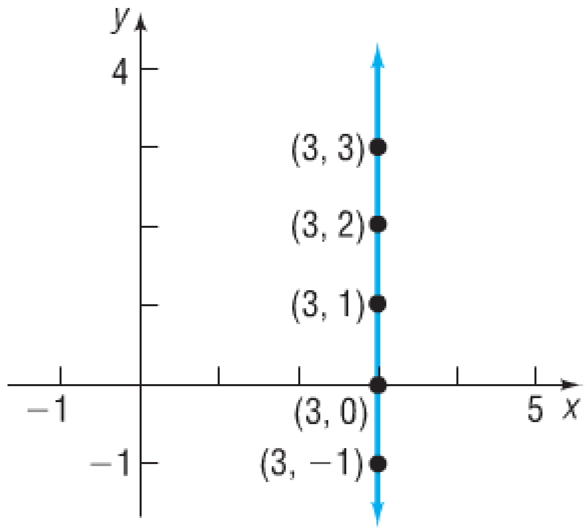
Slope-Intercept Form of a Line

If the equation of a non-vertical line in x and y is solved for y, the result is an equation of the form $y = mx + b$. The constant m is the slope of the line, and the line crosses the y-axis at b; that is, the y-intercept of the line is $(0, b)$. If the variable x does not appear in the equation, the slope is 0 and the equation is simply of the form $y = b$.

Use the slope-intercept form of the line to graph the equation $-5x + 4y = 12$.

Find the equation of the line that passes through the point $(0, -2)$ and has a slope of $\dfrac{-1}{3}$. Then graph the line.

Graph the equation: $x = 3$



Point Slope form of a line

Point-Slope Form of a Line

Given an ordered pair (x_1, y_1) and a real number m, an equation for the line passing through the point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$. Note that m, x_1, and y_1 are all constants, and that x and y are variables. Note also that since the line, by definition, has slope m, vertical lines cannot be described in this form.

EXAMPLE 6 **Finding the Equation of a Horizontal Line**

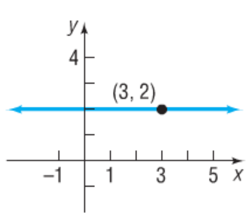
Find an equation of the horizontal line containing the point (3, 2).

Solution Because all the y-values are equal on a horizontal line, the slope of a horizontal line is 0. To get an equation, we use the point–slope form with $m = 0$, $x_1 = 3$, and $y_1 = 2$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 0 \cdot (x - 3) \qquad m = 0,\ x_1 = 3, \text{ and } y_1 = 2 \\ y - 2 &= 0 \\ y &= 2 \end{aligned}$$

See Figure 37 for the graph.

Figure 37



Find the equation, in slope-intercept form, of the line that passes through the point (–2, –1) with slope –4.

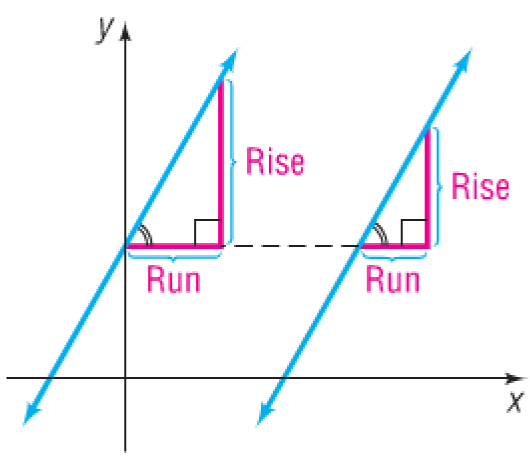
Find the equation, in slope-intercept form, of the line that passes through the two points (7, –3) and (5, 1).

2.4 Parallel and Perpendicular lines

THEOREM Criterion for Parallel Lines

Two nonvertical lines are parallel if and only if their slopes are equal and they have different y-intercepts.

Figure 43



EXAMPLE 10 Showing That Two Lines Are Parallel

Show that the lines given by the following equations are parallel:

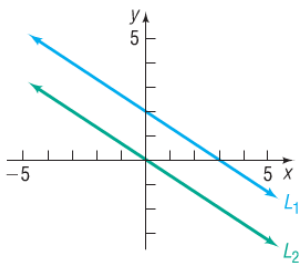
$$L_1:\quad 2x + 3y = 6, \qquad L_2:\quad 4x + 6y = 0$$

Solution To determine whether these lines have equal slopes and different y-intercepts, write each equation in slope–intercept form:

$$\begin{aligned} L_1:\quad 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= -\frac{2}{3}x + 2 \end{aligned} \qquad\qquad \begin{aligned} L_2:\quad 4x + 6y &= 0 \\ 6y &= -4x \\ y &= -\frac{2}{3}x \end{aligned}$$

$$\text{Slope} = -\frac{2}{3};\ y\text{-intercept} = 2 \qquad\qquad \text{Slope} = -\frac{2}{3};\ y\text{-intercept} = 0$$

Figure 44



Because these lines have the same slope, $-\frac{2}{3}$, but different y-intercepts, the lines are parallel. See Figure 44.

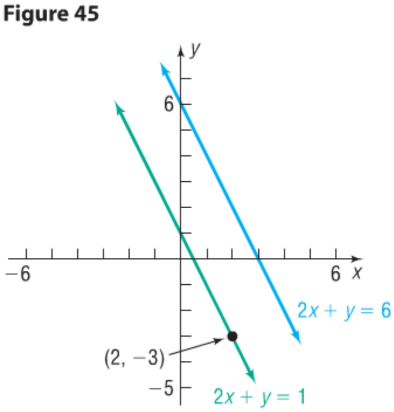
EXAMPLE 11 Finding a Line That Is Parallel to a Given Line

Find an equation for the line that contains the point $(2, -3)$ and is parallel to the line $2x + y = 6$.

Solution Since the two lines are to be parallel, the slope of the line that we seek equals the slope of the line $2x + y = 6$. Begin by writing the equation of the line $2x + y = 6$ in slope–intercept form.

$$\begin{aligned} 2x + y &= 6 \\ y &= -2x + 6 \end{aligned}$$

Figure 45



The slope is -2. Since the line that we seek also has slope -2 and contains the point $(2, -3)$, use the point–slope form to obtain its equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point–slope form} \\ y - (-3) &= -2(x - 2) && m = -2,\ x_1 = 2,\ y_1 = -3 \\ y + 3 &= -2x + 4 && \text{Simplify.} \\ y &= -2x + 1 && \text{Slope–intercept form} \\ 2x + y &= 1 && \text{General form} \end{aligned}$$

This line is parallel to the line $2x + y = 6$ and contains the point $(2, -3)$. See Figure 45.

Now Work PROBLEM 59

Slope of Perpendicular Lines

If m_1 and m_2 represent the slopes of two perpendicular lines, neither of which is vertical,

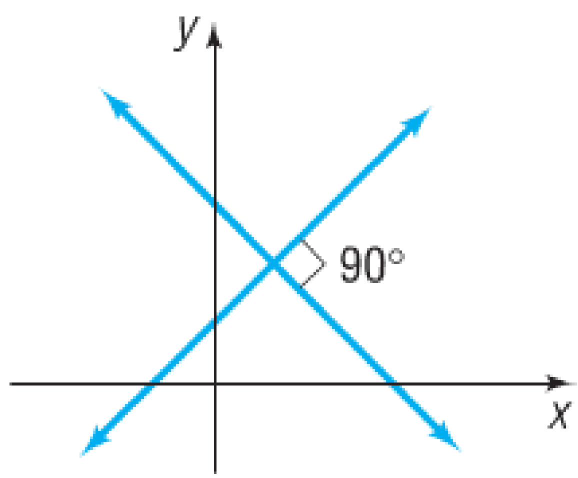
$$m_1 = \frac{-1}{m_2} \quad \text{and} \quad m_2 = \frac{-1}{m_1}$$

if one of the two perpendicular lines is vertical, the other is horizontal, and the slopes are, respectively, undefined and zero.

THEOREM

Criterion for Perpendicular Lines

Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.



Finding the Equation of a Line Perpendicular to a Given Line

Find an equation of the line that contains the point $(1, -2)$ and is perpendicular to the line $x + 3y = 6$. Graph the two lines.

First write the equation of the given line in slope–intercept form to find its slope.

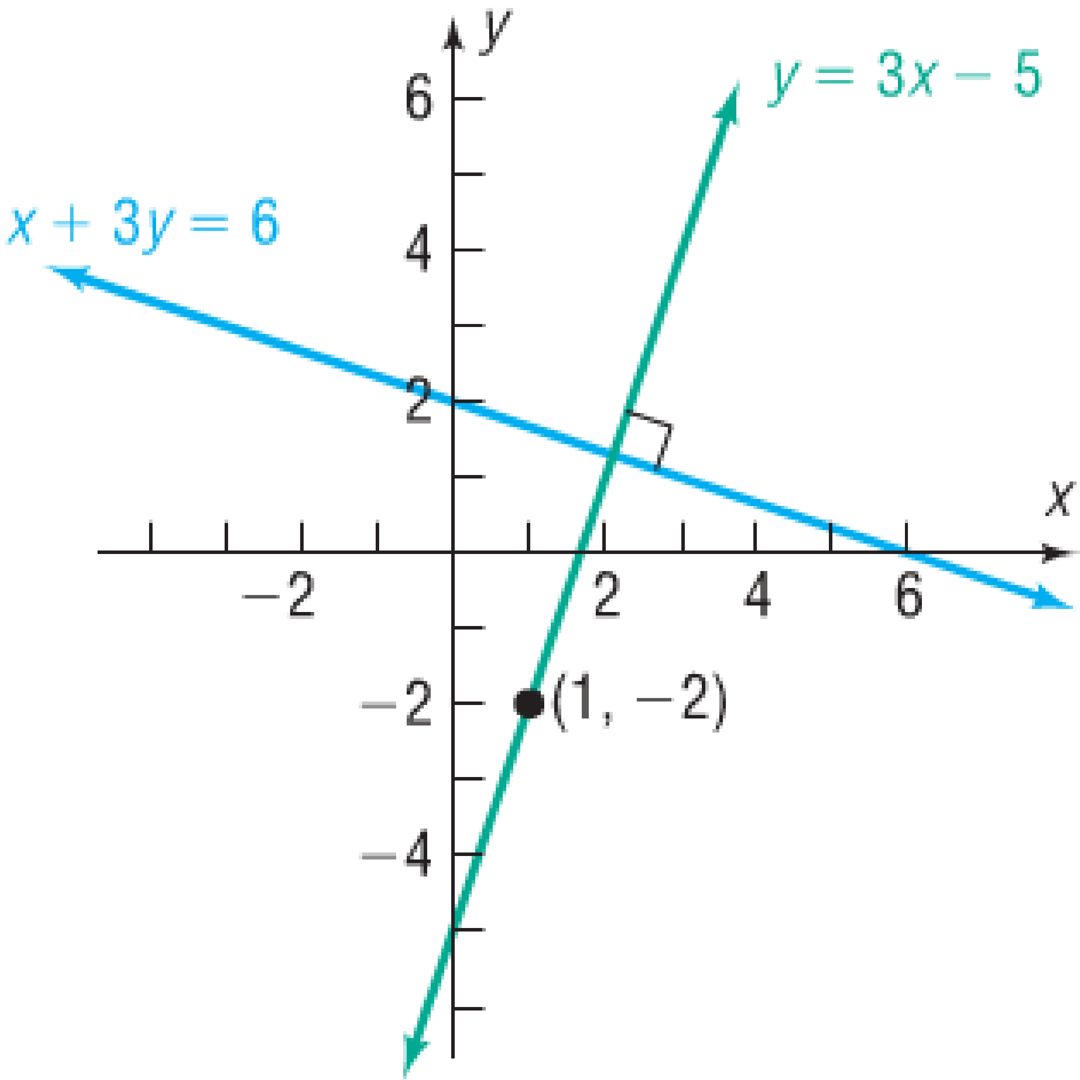
$$\begin{aligned} x + 3y &= 6 \\ 3y &= -x + 6 && \text{Proceed to solve for } y. \\ y &= -\frac{1}{3}x + 2 && \text{Place in the form } y = mx + b. \end{aligned}$$

The given line has slope $-\frac{1}{3}$. Any line perpendicular to this line will have slope 3. Because we require the point $(1, -2)$ to be on this line with slope 3, use the point–slope form of the equation of a line.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point–slope form} \\ y - (-2) &= 3(x - 1) && m = 3, x_1 = 1, y_1 = -2 \end{aligned}$$

To obtain other forms of the equation, proceed as follows:

$$\begin{aligned} y + 2 &= 3x - 3 && \text{Simplify.} \\ y &= 3x - 5 && \text{Slope–intercept form} \\ 3x - y &= 5 && \text{General form} \end{aligned}$$


y
x
$y = 3x - 5$
$x + 3y = 6$
6
4
2
−2
−4
−2
2
4
6
(1, −2)

2.5

Linear Inequalities in Two Variables

A linear inequality in the two variables x and y is an inequality that can be written in the form

$$ax + by < c,\quad ax + by > c,\quad ax + by \le c,\quad \text{or}\quad ax + by \ge c,$$

where a, b, and c are constants and a and b are not both 0.

Step 1: **Graph the line** in $\mathbb{R}^2$ that results from replacing the inequality symbol with =. Make the line solid if the inequality symbol is $\le$ or $\ge$ and dashed if the symbol is $<$ or $>$. A solid line indicates that points on the line will be included in the eventual solution set, while a dashed line indicates that points on the line are to be excluded from the solution set.

Step 2: Determine which of the half-planes defined by the boundary line solves the inequality by substituting a **test point** from one of the two half-planes into the inequality. If the resulting numerical statement is true, all the points in the same half-plane as the test point solve the inequality. Otherwise, the points in the other half-plane solve the inequality. **Shade in the half-plane** that solves the inequality.

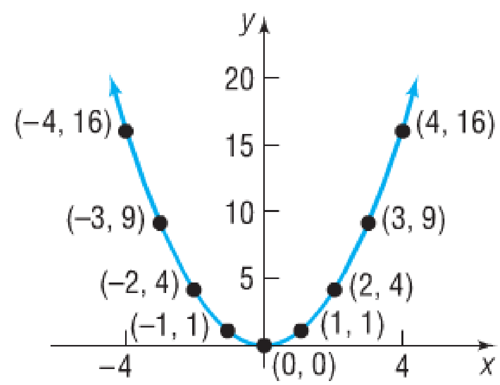
Solve the following linear inequalities by graphing their solution sets.

a. $3x - 4y < -12$ **b.** $x + y \le 0$ **c.** $y > \dfrac{5}{2}$

Graph the equation: $y = x^2$

x	$y = x^2$	(x, y)
-4	16	$(-4, 16)$
-3	9	$(-3, 9)$
-2	4	$(-2, 4)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	4	$(2, 4)$
3	9	$(3, 9)$
4	16	$(4, 16)$

Figure 13



Find the x-intercept(s) and the y-intercept(s) of the graph of $y = x^2 - 4$. Then graph $y = x^2 - 4$ by plotting points.

To find the x-intercept(s), let $y = 0$ and obtain the equation

$$x^2 - 4 = 0 \quad y = x^2 - 4 \text{ with } y = 0$$
$$(x + 2)(x - 2) = 0 \quad \text{Factor.}$$
$$x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{Zero-Product Property}$$
$$x = -2 \quad \text{or} \quad x = 2 \quad \text{Solve.}$$

The equation has two solutions, -2 and 2. The x-intercepts are -2 and 2.

To find the y-intercept(s), let $x = 0$ in the equation.

$$y = x^2 - 4$$
$$= 0^2 - 4 = -4$$

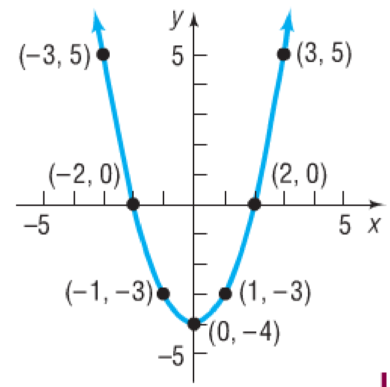
The y-intercept is -4.

Since $x^2 \geq 0$ for all x, we deduce from the equation $y = x^2 - 4$ that $y \geq -4$ for all x. This information, the intercepts, and the points from Table 2 enable us to graph $y = x^2 - 4$. See Figure 16.

Table 2

x	$y = x^2 - 4$	(x, y)
−3	5	(−3, 5)
−1	−3	(−1, −3)
1	−3	(1, −3)
3	5	(3, 5)

Figure 16



Point out the y intercept and how it shifted from y= x^2

Graph the equation $y = x^3$ by plotting points. Find any intercepts and check for symmetry first.

First, find the intercepts. When $x = 0$, then $y = 0$; and when $y = 0$, then $x = 0$. The origin $(0, 0)$ is the only intercept. Now test for symmetry.

x-Axis: Replace y by $-y$. Since $-y = x^3$ is not equivalent to $y = x^3$, the graph is not symmetric with respect to the x-axis.

y-Axis: Replace x by $-x$. Since $y = (-x)^3 = -x^3$ is not equivalent to $y = x^3$, the graph is not symmetric with respect to the y-axis.

Origin: Replace x by $-x$ and y by $-y$. Since $-y = (-x)^3 = -x^3$ is equivalent to $y = x^3$ (multiply both sides by -1), the graph is symmetric with respect to the origin.

To graph $y = x^3$, we use the equation to obtain several points on the graph. Because of the symmetry, we only need to locate points on the graph for which $x \geq 0$. See Table 3. Since $(1, 1)$ is on the graph, and the graph is symmetric with respect to the origin, the point $(-1, -1)$ is also on the graph. Plot the points from Table 3 and use the symmetry. Figure 21 shows the graph.

y=x^3

x	$y = x^3$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)
3	27	(3, 27)

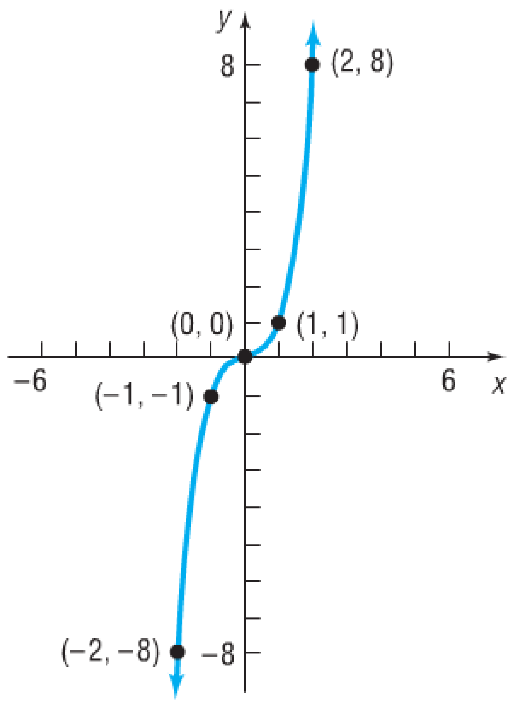
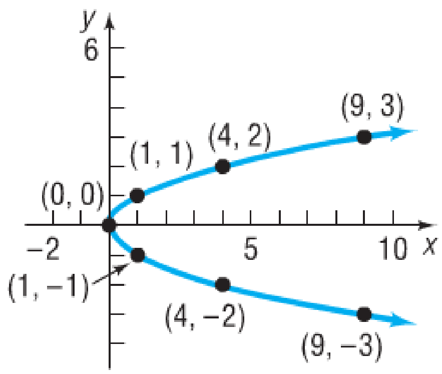
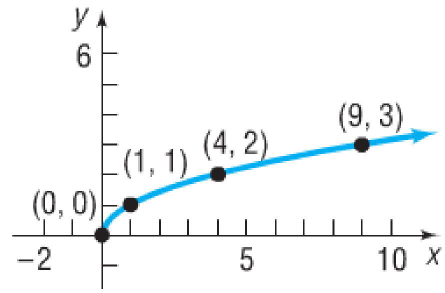


Figure 22
$x = y^2$



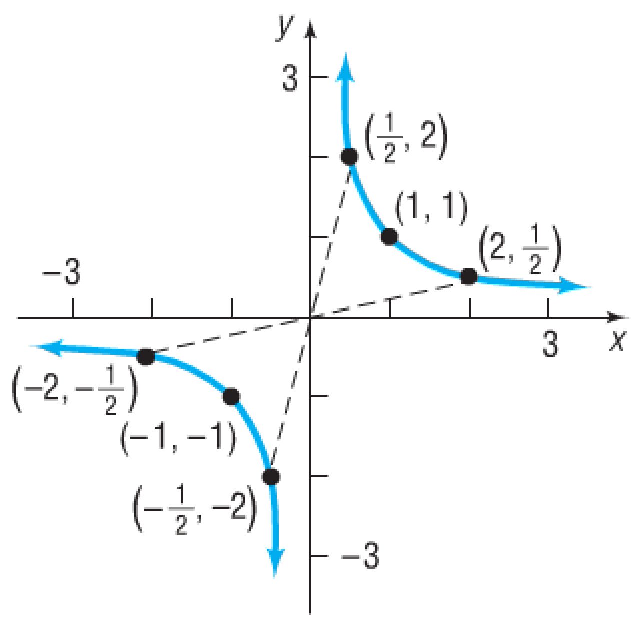
y=Sqrt of x

Figure 23
$y = \sqrt{x}$



Graphing the Equation $y = \frac{1}{x}$

x	$y = \frac{1}{x}$	(x, y)
$\frac{1}{10}$	10	$\left(\frac{1}{10}, 10\right)$
$\frac{1}{3}$	3	$\left(\frac{1}{3}, 3\right)$
$\frac{1}{2}$	2	$\left(\frac{1}{2}, 2\right)$
1	1	$(1, 1)$
2	$\frac{1}{2}$	$\left(2, \frac{1}{2}\right)$
3	$\frac{1}{3}$	$\left(3, \frac{1}{3}\right)$
10	$\frac{1}{10}$	$\left(10, \frac{1}{10}\right)$

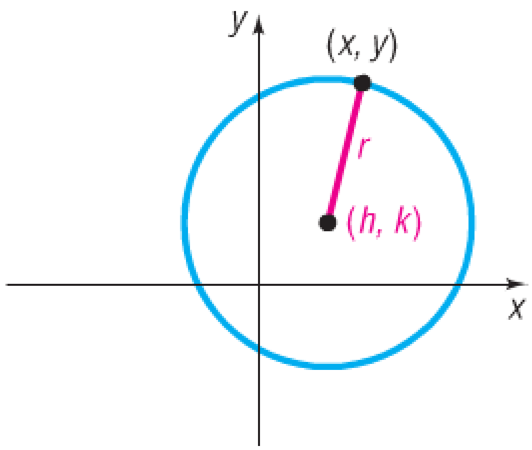


2.6 Circles

Standard Form of a Circle

The **standard form** of the equation for a circle of radius r with center (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2.$$



THEOREM

The standard form of an equation of a circle of radius r with center at the origin $(0, 0)$ is

$$\boxed{x^2 + y^2 = r^2}$$

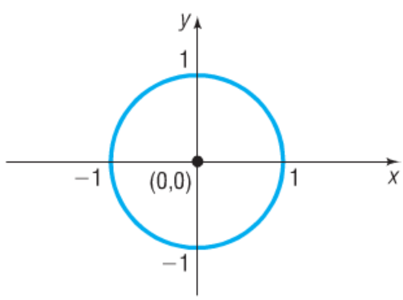
DEFINITION

If the radius $r = 1$, the circle whose center is at the origin is called the **unit circle** and has the equation

$$\boxed{x^2 + y^2 = 1}$$

See Figure 50. Notice that the graph of the unit circle is symmetric with respect to the x-axis, the y-axis, and the origin.

Figure 50
Unit circle $x^2 + y^2 = 1$



Writing the Standard Form of the Equation of a Circle

Write the standard form of the equation of the circle with radius 5 and center $(-3, 6)$.

Using equation (1) and substituting the values $r = 5$, $h = -3$, and $k = 6$, we have

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x + 3)^2 + (y - 6)^2 = 25$$

Find the standard form of the equation for the circle with radius 3 and center (–2, 7).

Graphing
Circles

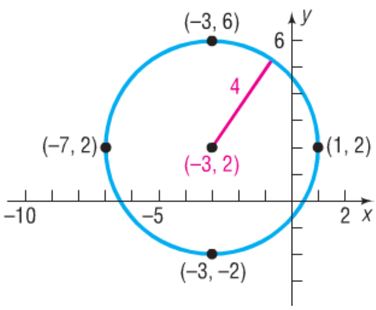
Graph the equation: $(x+3)^2 + (y-2)^2 = 16$

Solution Since the equation is in the form of equation (1), its graph is a circle. To graph the equation, compare the given equation to the standard form of the equation of a circle. The comparison yields information about the circle.

$$\begin{aligned}(x+3)^2 + (y-2)^2 &= 16\\ (x-(-3))^2 + (y-2)^2 &= 4^2\\ (x-h)^2 + (y-k)^2 &= r^2\end{aligned}$$

We see that $h = -3$, $k = 2$, and $r = 4$. The circle has center $(-3, 2)$ and a radius of 4 units. To graph this circle, first plot the center $(-3, 2)$. Since the radius is 4, we can locate four points on the circle by plotting points 4 units to the left, to the right, up, and down from the center. These four points can then be used as guides to obtain the graph. See Figure 51.

Figure 51



Sketch the graph of the circle defined by

$$(x-2)^2 + (y+3)^2 = 4.$$

3 Work with the General Form of the Equation of a Circle

If we eliminate the parentheses from the standard form of the equation of the circle given in Example 2, we get

$$\begin{aligned}(x+3)^2 + (y-2)^2 &= 16\\ x^2 + 6x + 9 + y^2 - 4y + 4 &= 16\end{aligned}$$

which, upon simplifying, is equivalent to

$$x^2 + y^2 + 6x - 4y - 3 = 0 \qquad \textbf{(2)}$$

It can be shown that any equation of the form

$$x^2 + y^2 + ax + by + c = 0$$

has a graph that is a circle, or a point, or has no graph at all. For example, the graph of the equation $x^2 + y^2 = 0$ is the single point $(0, 0)$. The equation $x^2 + y^2 + 5 = 0$, or $x^2 + y^2 = -5$, has no graph, because sums of squares of real numbers are never negative.

DEFINITION

When its graph is a circle, the equation

$$\boxed{x^2 + y^2 + ax + by + c = 0}$$

is referred to as the **general form of the equation of a circle.**

EXAMPLE 4 Graphing a Circle Whose Equation Is in General Form

Graph the equation $x^2 + y^2 + 4x - 6y + 12 = 0$

Solution Group the terms involving x, group the terms involving y, and put the constant on the right side of the equation. The result is

$$(x^2 + 4x) + (y^2 - 6y) = -12$$

Next, complete the square of each expression in parentheses. Remember that any number added on the left side of the equation must also be added on the right.

$$(x^2 + 4x + \underbrace{4}_{\left(\frac{4}{2}\right)^2 = 4}) + (y^2 - 6y + \underbrace{9}_{\left(\frac{-6}{2}\right)^2 = 9}) = -12 + 4 + 9$$

$$(x + 2)^2 + (y - 3)^2 = 1 \quad \text{Factor.}$$

This equation is the standard form of the equation of a circle with radius 1 and center $(-2, 3)$.

To graph the equation use the center $(-2, 3)$ and the radius 1. See Figure 52.

Figure 52