Ratios of polynomials are called rational functions. They include:

$$R(x) = \frac{x^2 - 4}{x^2 + x + 1} \qquad F(x) = \frac{x^3}{x^2 - 4} \qquad G(x) = \frac{3x^2}{x^4 - 1}$$

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

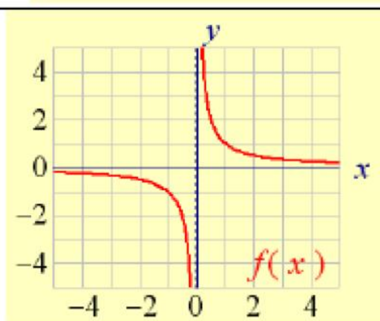
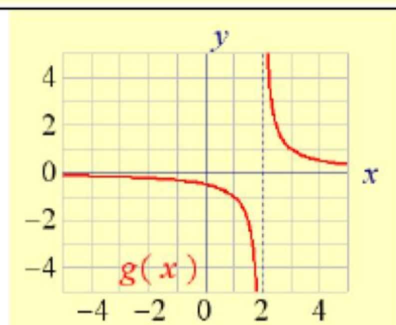
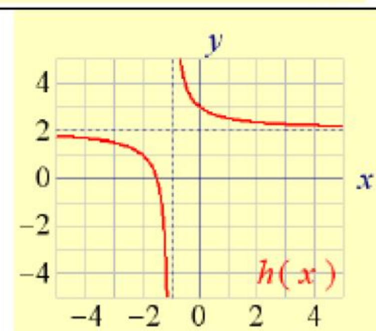
where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Rational Functions

A **rational function** is a function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are both polynomial functions and $q(x) \neq 0$. Of course, even though q is not allowed to be identically zero, there will often be values of x for which $q(x)$ is zero, and at these values the fraction is undefined. Consequently, the **domain of** f is the set $\{ x \mid q(x) \neq 0 \}$.

		
$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{x-2}$	$h(x) = \frac{1}{x+1} + 2$ $\frac{1}{x+1} + \frac{2x+2}{x+1} = \frac{2x+3}{x+1}$

.

Graphing y = 1 / x^2 and transformations

Graphing $y = \frac{1}{x^2}$

Graphing $y = \frac{1}{x^2}$

Analyze the graph of $H(x) = \frac{1}{x^2}$.

The domain of $H(x) = \frac{1}{x^2}$ is the set of all real numbers x except 0. The graph has no y-intercept, because x can never equal 0. The graph has no x-intercept because the equation $H(x) = 0$ has no solution. Therefore, the graph of H will not cross or touch either of the coordinate axes. Because

$$H(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = H(x)$$

H is an even function, so its graph is symmetric with respect to the y-axis.

Table 9 shows the behavior of $H(x) = \frac{1}{x^2}$ for selected positive numbers x. (We will use symmetry to obtain the graph of H when $x < 0$.) From the first three rows of Table 9, we see that, as the values of x approach (get closer to) 0, the values of $H(x)$ become larger and larger positive numbers, so H is unbounded in the positive direction. We use limit notation, $\lim_{x \to 0} H(x) = \infty$, read "the limit of $H(x)$ as x approaches zero equals infinity," to mean that $H(x) \to \infty$ as $x \to 0$.

Look at the last four rows of Table 9. As $x \to \infty$, the values of $H(x)$ approach 0 (the end behavior of the graph). In calculus, this is symbolized by writing $\lim_{x \to \infty} H(x) = 0$. Figure 28 shows the graph. Notice the use of red dashed lines to convey the ideas discussed above.

Figure 28

$H(x) = \frac{1}{x^2}$

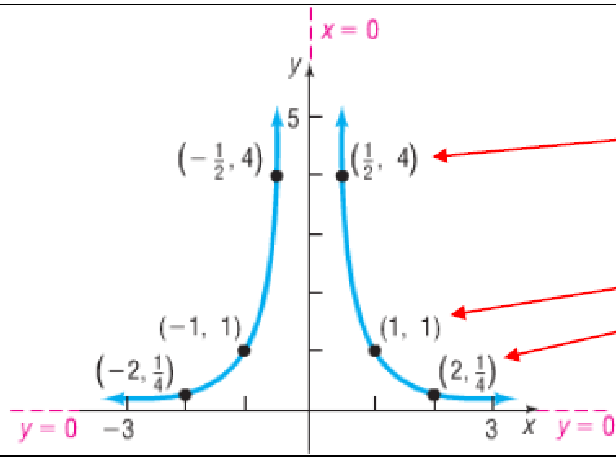


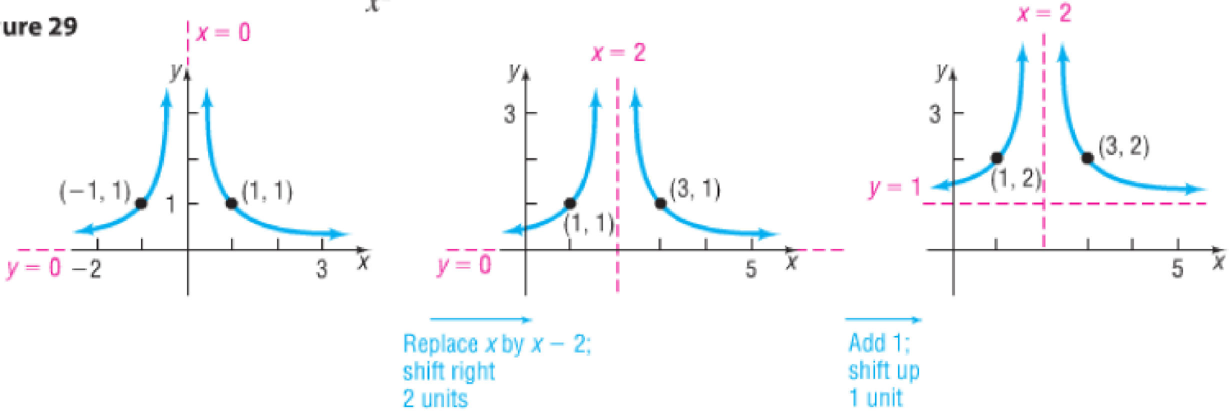
Table 9

x	$H(x) = \frac{1}{x^2}$
$\frac{1}{2}$	4
$\frac{1}{100}$	10,000
$\frac{1}{10,000}$	100,000,000
1	1
2	$\frac{1}{4}$
100	$\frac{1}{10,000}$
10,000	$\frac{1}{100,000,000}$

Graph the rational function: $R(x) = \dfrac{1}{(x-2)^2} + 1$

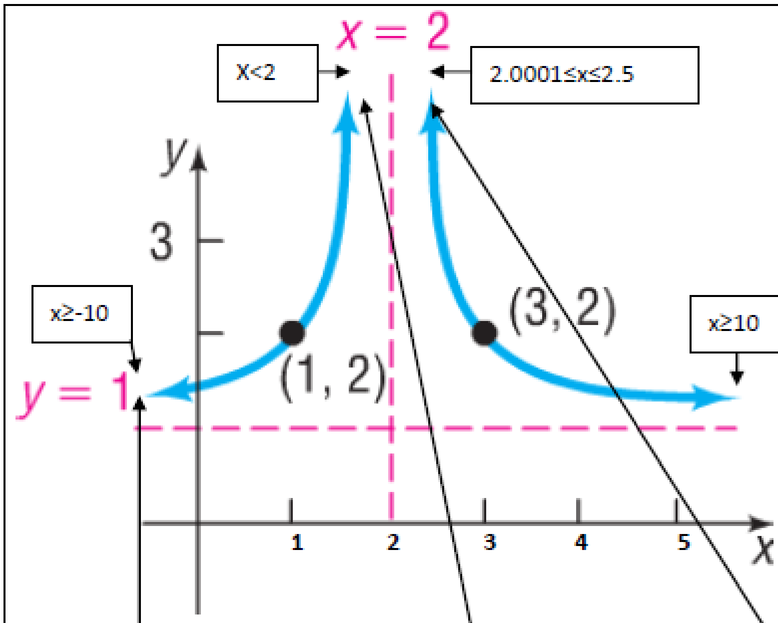
Solution The domain of R is the set of all real numbers except $x = 2$. To graph R, start with the graph of $y = \dfrac{1}{x^2}$. See Figure 29 for the steps.

Figure 29



(a) $y = \dfrac{1}{x^2}$ (b) $y = \dfrac{1}{(x-2)^2}$ (c) $y = \dfrac{1}{(x-2)^2} + 1$

Asymptotes – a line approached by a curve in the limit as the curve approaches infinity.



$$R(x) = y = \frac{1}{(x-2)^2} + 1$$

Vertical line x =2 and horizontal line y =1 in the above figure.

HORIZONTAL ASYMPTOTE

x	R(x)
10	1.0156
100	1.0001
1000	1.000001
10,000	1.00000001

(a)

The higher the value of x = 10, 100, 1000, 10000 the R(x) or y = keeps reaching the value of 1 in the positive direction.

Limits are "x" to positive infinity

Horizontal = x→+infinity R(x =1)

HORIZONTAL ASYMPTOTE

x	R(x)
−10	1.0069
−100	1.0001
−1000	1.000001
−10,000	1.00000001

(b)

The higher the value of x = -10, -100, -1000, -10000 the

R(x) **or y** = keeps reaching the value of 1 in the negative direction.

Limits are "x" to negative infinity

R(x)=1 or y = 1

Horizontal = x→-infinity

VERTICAL ASYMPTOTE

x	R(x)
1.5	5
1.9	101
1.99	10,001
1.999	1,000,001
1.9999	100,000,001

(c)

x=2 is not part of the domain but it is important to see the graphs behavior as it approaches x =2

As "x"< 2

the value for "y" or R(x) increases to infinity.

x =1.99 results y= 100,000,001

$x \to 2^-$

Vertical = "y" or R(x) →+inf

VERTICAL ASYMPTOTE

x	R(x)
2.5	5
2.1	101
2.01	10,001
2.001	1,000,001
2.0001	100,000,001

(d)

As "x"> 2

the value for "y" or R(x) increases to infinity.

x = results y= 100,000,001

To see the effect better

2.0001 ≤ "x" ≤ 2.5

the value for $x \to 2^+$

Vertical = "y" or R(x) →+inf

Vertical & horizontal (oblique) asymptotes of a rational function

Vertical Asymptotes

The vertical line $x = c$ is a **vertical asymptote** of a function f if $f(x)$ increases in magnitude without bound as x approaches c. Examples of vertical asymptotes appear in Figure 2. The graph of a rational function cannot intersect a vertical asymptote.

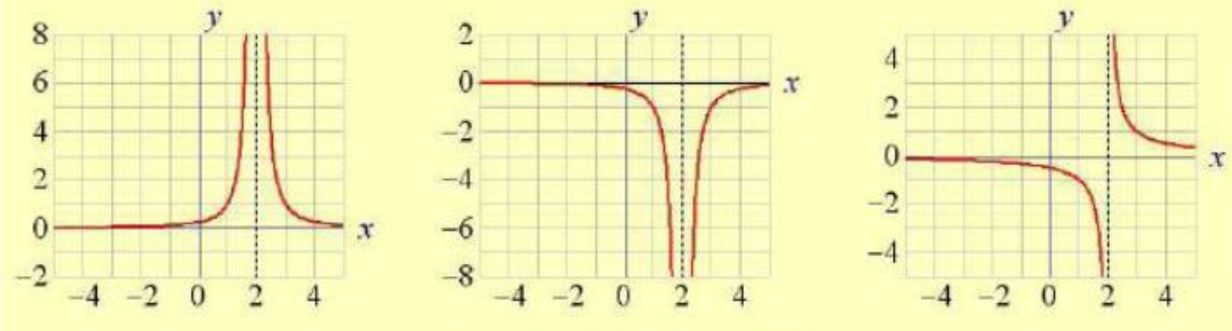


Figure 2: Vertical Asymptotes

Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \dfrac{x+3}{x-1}$ (b) $R(x) = \dfrac{x}{x^2-4}$

(c) $H(x) = \dfrac{x^2}{x^2+1}$ (d) $G(x) = \dfrac{x^2-9}{x^2+4x-21}$

(a) F is in lowest terms and the only zero of the denominator is 1. The line $x = 1$ is the vertical asymptote of the graph of F.

(b) R is in lowest terms and the zeros of the denominator $x^2 - 4$ are -2 and 2. The lines $x = -2$ and $x = 2$ are the vertical asymptotes of the graph of R.

(c) H is in lowest terms and the denominator has no real zeros, because the equation $x^2 + 1 = 0$ has no real solutions. The graph of H has no vertical asymptotes.

(d) Factor the numerator and denominator of $G(x)$ to determine if it is in lowest terms.

$$G(x) = \frac{x^2-9}{x^2+4x-21} = \frac{(x+3)(x-3)}{(x+7)(x-3)} = \frac{x+3}{x+7} \qquad x \neq 3$$

The only zero of the denominator of $G(x)$ in lowest terms is -7. The line $x = -7$ is the only vertical asymptote of the graph of G.

Finding a Horizontal Asymptote

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

n Since the degree of the numerator, 1, is less than the degree of the denominator, 2, the rational function R is proper. The line $y = 0$ is a horizontal asymptote of the graph of R.

To see why $y = 0$ is a horizontal asymptote of the function R in Example 5, we investigate the behavior of R as $x \to -\infty$ and $x \to \infty$. When $|x|$ is very large, the numerator of R, which is $x - 12$, can be approximated by the power function $y = x$, while the denominator of R, which is $4x^2 + x + 1$, can be approximated by the power function $y = 4x^2$. Applying these ideas to $R(x)$, we find

$$R(x) = \frac{x - 12}{4x^2 + x + 1} \underset{\uparrow}{\approx} \frac{x}{4x^2} = \frac{1}{4x} \underset{\uparrow}{\to} 0$$

For $|x|$ very large As $x \to -\infty$ or $x \to \infty$

This shows that the line $y = 0$ is a horizontal asymptote of the graph of R.

If a rational function $R(x) = \frac{p(x)}{q(x)}$ is **improper,** that is, if the degree of the numerator is greater than or equal to the degree of the denominator, we use long division to write the rational function as the sum of a polynomial $f(x)$ (the quotient) plus a proper rational function $\frac{r(x)}{q(x)}$ ($r(x)$ is the remainder). That is, we write

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

where $f(x)$ is a polynomial and $\frac{r(x)}{q(x)}$ is a proper rational function. Since $\frac{r(x)}{q(x)}$ is proper, $\frac{r(x)}{q(x)} \to 0$ as $x \to -\infty$ or as $x \to \infty$. As a result,

$$R(x) = \frac{p(x)}{q(x)} \to f(x) \qquad \text{as } x \to -\infty \text{ or as } x \to \infty$$

The possibilities are listed next.

1. If $f(x) = b$, a constant, the line $y = b$ is a horizontal asymptote of the graph of R.

Slanted 45 degrees

2. If $f(x) = ax + b$, $a \neq 0$, the line $y = ax + b$ is an oblique asymptote of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

We illustrate each of the possibilities in Examples 6, 7, and 8.

EX 7

Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

Since the degree of the numerator, 2, equals the degree of the denominator, 2, the rational function R is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r} 2 \\ 4x^2 - 1 \overline{)8x^2 - x + 2} \\ \underline{8x^2 \qquad - 2} \\ - x + 4 \end{array}$$

As a result,

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1} = 2 + \frac{-x + 4}{4x^2 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{-x + 4}{4x^2 - 1} \approx \frac{-x}{4x^2} = \frac{-1}{4x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $R(x) \to 2$. We conclude that $y = 2$ is a horizontal asymptote of the graph.

In Example 7, notice that the quotient 2 obtained by long division is the quotient of the leading coefficients of the numerator polynomial and the denominator polynomial $\left(\frac{8}{4}\right)$. This means that we can avoid the long division process for rational functions where the numerator and denominator *are of the same degree* and conclude that the quotient of the leading coefficients will give us the horizontal asymptote.

$$f(x) = \frac{6x^2 - 3x + 2}{3x^2 + 5x - 17}$$

In symbols, $f(x) \to 2$ as $x \to \pm\infty$.

If the degree are equal then divide the coefficient to obtain the horizontal asymptote.

EX 8

Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

n

Since the degree of the numerator, 5, is greater than the degree of the denominator, 3, the rational function G is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r|l} & 2x^2 - 1 \\ \hline x^3 - 1 & 2x^5 - x^3 \qquad\quad + 2 \\ & \underline{2x^5 \qquad - 2x^2} \\ & \quad -x^3 + 2x^2 + 2 \\ & \quad \underline{-x^3 \qquad\quad + 1} \\ & \qquad\quad 2x^2 + 1 \end{array}$$

As a result,

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1} = 2x^2 - 1 + \frac{2x^2 + 1}{x^3 - 1}$$

Then, as $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 + 1}{x^3 - 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $G(x) \to 2x^2 - 1$. We conclude that, for large values of $|x|$, the graph of G approaches the graph of $y = 2x^2 - 1$. That is, the graph of G will look like the graph of $y = 2x^2 - 1$ as $x \to -\infty$ or $x \to \infty$. Since $y = 2x^2 - 1$ is not a linear function, G has no horizontal or oblique asymptote.

$$f(x) = \frac{3x^5 - 2x^3 + 7x^2 - 1}{4x^3 + 19x^2 - 3x + 5}$$

If the degree in the numerator is greater than the degree in the denominator and it is greater than 1 then the function is not oblique or horizontal.

EX 6

Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Since the degree of the numerator, 4, is greater than the degree of the denominator, 3, the rational function H is improper. To find a horizontal or oblique asymptote, we use long division.

$$\begin{array}{r|l} & 3x + 3 \\ \hline x^3 - x^2 + 1 & 3x^4 \qquad\qquad - x^2 \\ & \underline{3x^4 - 3x^3 \qquad\quad + 3x} \\ & \qquad 3x^3 - x^2 - 3x \\ & \qquad \underline{3x^3 - 3x^2 \qquad + 3} \\ & \qquad\qquad 2x^2 - 3x - 3 \end{array}$$

As a result,

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1} = 3x + 3 + \frac{2x^2 - 3x - 3}{x^3 - x^2 + 1}$$

As $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 - 3x - 3}{x^3 - x^2 + 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

As $x \to -\infty$ or as $x \to \infty$, we have $H(x) \to 3x + 3$. We conclude that the graph of the rational function H has an oblique asymptote $y = 3x + 3$.

$$f(x) = \frac{7x^3 + 2x - 1}{x^2 + 4x}$$

Since the degree in the numerator is greater than the numerator then do long division.

$-3x^3 + 7x^2 + 8x + 1$ by $x - 3$

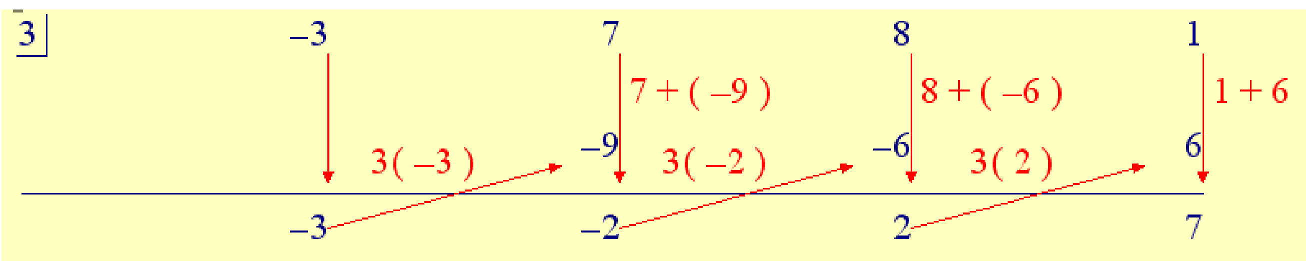
$$
\begin{array}{r|rrrrrrr}
 & & -3x^2 & -2x & +2 & & & \\
\hline
x-3 & -3x^3 & +7x^2 & +8x & +1 & & & \\
 & -(-3x^3 & +9x^2) & & & & & \\
\hline
 & & -2x^2 & +8x & +1 & & & \\
 & & -(-2x^2 & +6x) & & & & \\
\hline
 & & & 2x & +1 & & & \\
 & & & -(2x & -6) & & & \\
\hline
 & & & & 7 & & &
\end{array}
$$

Synthetic Division requires the divisor to be in the format of "(x-k)"

Take the opposite of (x – 3)←→ (x - (+3))

3	–3	7	8	1
		–9	–6	6
	–3	–2	2	7

Take the opposite of (x – 3) ←→ (x - (+3))



Example. Use synthetic division to perform the indicated operation: $\dfrac{x^4 - 3x^3 - 5x^2 + 2x - 18}{x + 2}$

Solution. The divisor, x + 2, must be written as a difference, x – (-2), to determine that the divider is –2.

-2	1	-3	-5	2	-18
		-2	10	-10	16
	1	-5	5	-8	-2

$$\frac{x^4 - 3x^3 - 5x^2 + 2x - 16}{x + 2} = x^3 - 5x^2 + 5x - 8 + \frac{-2}{x + 2}$$

f(x) = d(x) · q(x) + r(x)/d(x)

Dividend = Divisor · Quotient + Remainder/Divisor

Find the equation for the horizontal or oblique asymptote of the following functions.

a. $g(x) = \dfrac{x+1}{x^2+2x-15}$ **b.** $h(x) = \dfrac{x^3-27}{2x+2}$

Solutions:

a. Because the numerator has a degree less than the degree of the denominator of g, the asymptote is the horizontal line $y = 0$.

b. Because the degree of the numerator is greater than one plus the degree of the denominator, we know h has no oblique or horizontal asymptote.

a. $f(x) = \dfrac{x^3-x^2+4}{x+5}$ **Vertical** $x =$ ___ **horizontal/oblique** $y =$ ___

b. $g(x) = \dfrac{2x-3}{x^2-9}$ **Vertical** $x =$ ___ **horizontal/oblique** $y =$ ___

c. $h(x) = \dfrac{x^2+x-7}{2x^2-8}$ **Vertical** $x =$ ___ **horizontal/oblique** $y =$ ___

a. $f(x) = \dfrac{x^3-x^2+4}{x+5}$ **Vertical** $x = -5$ **horizontal/oblique** $y =$ *None*

If the degree in the numerator is greater than the degree in the denominator and it is greater than 1 then the function is not oblique or horizontal.

b. $g(x) = \dfrac{2x-3}{x^2-9}$ **Vertical** $x = 3, -3$ **horizontal/oblique** $y = 0$

"x" Vertical asymptote to make the denominator equal to zero.
The rational equation is proper because the degree for the numerator is less than the degree in the denominator.

c. $h(x) = \dfrac{x^2+x-7}{2x^2-8}$ **Vertical** $x = 2, -2$ **horizontal/oblique** $y = \dfrac{1}{2}$

The rational equation is improper because the degree for the numerator is equal to the degree in the denominator.
Same degrees divide coefficients for "y" horizontal asymptote
"x" Vertical asymptote to make the denominator equal to zero.

SUMMARY Finding a Horizontal or Oblique Asymptote of a Rational Function

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).
2. If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then R is improper. Here long division is used.
   (a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
   (b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.
   (c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for very large values of $|x|$, the graph of R will behave like the graph of the quotient.

Note: The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote.

Equations for Horizontal and Oblique Asymptotes

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function, where p is an n^{th} degree polynomial with leading coefficient a_n and q is an m^{th} degree polynomial with leading coefficient b_m.

Then:

1. If $n < m$, the horizontal line $y = 0$ (the x-axis) is the horizontal asymptote for f.
2. If $n = m$, the horizontal line $y = \frac{a_n}{b_m}$ is the horizontal asymptote for f.
3. If $n = m + 1$, the line $y = g(x)$ is an oblique asymptote for f, where g is the quotient polynomial obtained by dividing p by q (the remainder polynomial is irrelevant).
4. If $n > m + 1$, there is no straight-line horizontal or oblique asymptote for f.