CHAPTER 5 SULLIVAN 9th Ed

5.6 Exponential and Logarithmic Equations

Solve Logarithmic Equations

In Section 5.4 we solved logarithmic equations by changing a logarithmic expression to an exponential expression. That is, we used the definition of a logarithm:

$$y = \log_a x \quad \text{is equivalent to} \quad x = a^y \qquad a > 0, a \neq 1$$

For example, to solve the equation $\log_2(1 - 2x) = 3$, we write the logarithmic equation as an equivalent exponential equation $1 - 2x = 2^3$ and solve for x.

$$\begin{aligned} \log_2(1 - 2x) &= 3 \\ 1 - 2x &= 2^3 && \text{Change to an exponential statement.} \\ -2x &= 7 && \text{Simplify.} \\ x &= -\frac{7}{2} && \text{Solve.} \end{aligned}$$

You should check this solution for yourself.

For most logarithmic equations, some manipulation of the equation (usually using properties of logarithms) is required to obtain a solution. Also, to avoid extraneous solutions with logarithmic equations, we determine the domain of the variable first.

We begin with an example of a logarithmic equation that requires using the fact that a logarithmic function is a one-to-one function:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N \qquad M, N, \text{ and } a \text{ are positive and } a \neq 1.$$

Example 1

Solving a Logarithmic Equation

Solve: $2 \log_5 x = \log_5 9$

The domain of the variable in this equation is $x > 0$. Because each logarithm is to the same base, 5, we can obtain an exact solution as follows:

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$$2 \log_5 x = \log_5 9$$

$$\log_5 x^2 = \log_5 9 \qquad r \log_a M = \log_a M^r$$

$$x^2 = 9 \qquad \text{If } \log_a M = \log_a N, \text{ then } M = N.$$

$$x = 3 \quad \text{or} \quad x = -3$$

Recall that the domain of the variable is $x > 0$. Therefore, -3 is extraneous and we discard it.

✓Check: $2 \log_5 3 \stackrel{?}{=} \log_5 9$

$$\log_5 3^2 \stackrel{?}{=} \log_5 9 \qquad r \log_a M = \log_a M^r$$

$$\log_5 9 = \log_5 9$$

The solution set is $\{3\}$.

Often we need to use one or more properties of logarithms to rewrite the equation as a single logarithm. In the next example we employ the log of a product property to solve a logarithmic equation.

Example 2

> **WARNING** A negative solution is not automatically extraneous. You must determine whether the potential solution causes the argument of any logarithmic expression in the equation to be negative. ■

Solve: $\log_5(x + 6) + \log_5(x + 2) = 1$

The domain of the variable requires that $x + 6 > 0$ and $x + 2 > 0$, so $x > -6$ and $x > -2$. This means any solution must satisfy $x > -2$. To obtain an exact solution, we need to express the left side as a single logarithm. Then we will change the equation to an equivalent exponential equation.

$$\log_5(x + 6) + \log_5(x + 2) = 1$$

$$\log_5[(x + 6)(x + 2)] = 1 \qquad \log_a M + \log_a N = \log_a(MN)$$

$$(x + 6)(x + 2) = 5^1 = 5 \qquad \text{Change to an exponential statement.}$$

$$x^2 + 8x + 12 = 5 \qquad \text{Simplify.}$$

$$x^2 + 8x + 7 = 0 \qquad \text{Place the quadratic equation in standard form.}$$

$$(x + 7)(x + 1) = 0 \qquad \text{Factor.}$$

$$x = -7 \quad \text{or} \quad x = -1 \qquad \text{Zero-Product Property}$$

Only $x = -1$ satisfies the restriction that $x > -2$, so $x = -7$ is extraneous. The solution set is $\{-1\}$, which you should check.

Example 3

Solve: $\ln x = \ln(x + 6) - \ln(x - 4)$

The domain of the variable requires that $x > 0$, $x + 6 > 0$, and $x - 4 > 0$. As a result, the domain of the variable here is $x > 4$. We begin the solution using the log of a difference property.

$$\ln x = \ln(x + 6) - \ln(x - 4)$$

$$\ln x = \ln\left(\frac{x + 6}{x - 4}\right) \qquad \ln M - \ln N = \ln\left(\frac{M}{N}\right)$$

$$x = \frac{x + 6}{x - 4} \qquad \text{If } \ln M = \ln N, \text{ then } M = N.$$

$$x(x - 4) = x + 6 \qquad \text{Multiply both sides by } x - 4.$$

$$x^2 - 4x = x + 6 \qquad \text{Simplify.}$$

$$x^2 - 5x - 6 = 0 \qquad \text{Place the quadratic equation in standard form.}$$

$$(x - 6)(x + 1) = 0 \qquad \text{Factor.}$$

$$x = 6 \quad \text{or} \quad x = -1 \qquad \text{Zero-Product Property}$$

Since the domain of the variable is $x > 4$, we discard -1 as extraneous. The solution set is $\{6\}$, which you should check.

WARNING In using properties of logarithms to solve logarithmic equations, avoid using the property $\log_a x^r = r \log_a x$, when r is even. The reason can be seen in this example:

Solve: $\log_3 x^2 = 4$

Solution: The domain of the variable x is all real numbers except 0.

(a) $\log_3 x^2 = 4$

$x^2 = 3^4 = 81$ Change to exponential form.

$x = -9$ or $x = 9$

(b) $\log_3 x^2 = 4$ $\quad \log_a x^r = r \log_a x$

$2 \log_3 x = 4$ $\quad$ Domain of variable is $x > 0$.

$\log_3 x = 2$

$x = 9$

Both -9 and 9 are solutions of $\log_3 x^2 = 4$ (as you can verify). The solution in part (b) does not find the solution -9 because the domain of the variable was further restricted due to the application of the property $\log_a x^r = r \log_a x$. ■

Solve Exponential Equations

In Sections 5.3 and 5.4, we solved exponential equations algebraically by expressing each side of the equation using the same base. That is, we used the one-to-one property of the exponential function:

$$\text{If } a^u = a^v, \quad \text{then } u = v \qquad a > 0, a \neq 1$$

For example, to solve the exponential equation $4^{2x+1} = 16$, notice that $16 = 4^2$ and apply the property above to obtain $2x + 1 = 2$, from which we find $x = \frac{1}{2}$.

For most exponential equations, we cannot express each side of the equation using the same base. In such cases, algebraic techniques can sometimes be used to obtain exact solutions.

Example 4

Solving Exponential Equations

Solve: (a) $2^x = 5$ (b) $8 \cdot 3^x = 5$

(a) Since 5 cannot be written as an integer power of 2 ($2^2 = 4$ and $2^3 = 8$), write the exponential equation as the equivalent logarithmic equation.

$$2^x = 5$$
$$x = \underbrace{\log_2 5 = \frac{\ln 5}{\ln 2}}_{\text{Change-of-Base Formula (10), Section 5.5}}$$

Alternatively, we can solve the equation $2^x = 5$ by taking the natural logarithm (or common logarithm) of each side. Taking the natural logarithm,

$$\begin{aligned} 2^x &= 5 \\ \ln 2^x &= \ln 5 && \text{If } M = N, \text{ then } \ln M = \ln N. \\ x \ln 2 &= \ln 5 && \ln M^r = r \ln M \\ x &= \frac{\ln 5}{\ln 2} && \text{Exact solution} \\ &\approx 2.322 && \text{Approximate solution} \end{aligned}$$

The solution set is $\left\{\frac{\ln 5}{\ln 2}\right\}$.

(b)
$$\begin{aligned} 8 \cdot 3^x &= 5 \\ 3^x &= \frac{5}{8} && \text{Solve for } 3^x. \end{aligned}$$

$$\begin{aligned} x = \log_3\left(\frac{5}{8}\right) &= \frac{\ln\left(\frac{5}{8}\right)}{\ln 3} && \text{Exact solution} \\ &\approx -0.428 && \text{Approximate solution} \end{aligned}$$

The solution set is $\left\{\frac{\ln\left(\frac{5}{8}\right)}{\ln 3}\right\}$.

Example 5

Solve: $5^{x-2} = 3^{3x+2}$

Because the bases are different, we first apply property (7), Section 5.5 (take the natural logarithm of each side), and then use a property of logarithms. The result is an equation in x that we can solve.

$$\begin{aligned}
5^{x-2} &= 3^{3x+2} \\
\ln 5^{x-2} &= \ln 3^{3x+2} && \text{If } M = N, \ln M = \ln N. \\
(x-2)\ln 5 &= (3x+2)\ln 3 && \ln M^r = r \ln M \\
(\ln 5)x - 2\ln 5 &= (3\ln 3)x + 2\ln 3 && \text{Distribute.} \\
(\ln 5)x - (3\ln 3)x &= 2\ln 3 + 2\ln 5 && \text{Place terms involving } x \text{ on the left.} \\
(\ln 5 - 3\ln 3)x &= 2(\ln 3 + \ln 5) && \text{Factor.} \\
x &= \frac{2(\ln 3 + \ln 5)}{\ln 5 - 3\ln 3} && \text{Exact solution} \\
&\approx -3.212 && \text{Approximate solution}
\end{aligned}$$

The solution set is $\left\{\dfrac{2(\ln 3 + \ln 5)}{\ln 5 - 3\ln 3}\right\}$.

Example 6

Solve: $4^x - 2^x - 12 = 0$

We note that $4^x = (2^2)^x = 2^{(2x)} = (2^x)^2$, so the equation is quadratic in form, and we can rewrite it as

$$(2^x)^2 - 2^x - 12 = 0 \qquad \text{Let } u = 2^x\text{; then } u^2 - u - 12 = 0.$$

Now we can factor as usual.

$$\begin{aligned}
&(2^x - 4)(2^x + 3) = 0 && (u-4)(u+3) = 0 \\
&2^x - 4 = 0 \quad \text{or} \quad 2^x + 3 = 0 && u - 4 = 0 \quad \text{or} \quad u + 3 = 0 \\
&2^x = 4 \qquad\qquad 2^x = -3 && u = 2^x = 4 \qquad u = 2^x = -3
\end{aligned}$$

The equation on the left has the solution $x = 2$, since $2^x = 4 = 2^2$; the equation on the right has no solution, since $2^x > 0$ for all x. The only solution is 2. The solution set is $\{2\}$.

Solve Logarithmic and Exponential Equations Using a Graphing Utility

The algebraic techniques introduced in this section to obtain exact solutions apply only to certain types of logarithmic and exponential equations. Solutions for other types are usually studied in calculus, using numerical methods. For such types, we can use a graphing utility to approximate the solution.