CHAPTER 5 SULLIVAN 9^{th} Ed

5.5 Properties and Applications of Logarithms

SUMMARY Properties of Logarithms

In the list that follows, a, b, M, N, and r are real numbers. Also, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Definition	$y = \log_a x$ means $x = a^y$	
Properties of logarithms	$\log_a 1 = 0$; $\log_a a = 1$	$\log_a M^r = r \log_a M$
	$a^{\log_a M} = M$; $\log_a a^r = r$	$a^x = e^{x \ln a}$
	$\log_a(MN) = \log_a M + \log_a N$	If $M = N$, then $\log_a M = \log_a N$.
	$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$	If $\log_a M = \log_a N$, then $M = N$.
Change-of-Base Formula	$\log_a M = \frac{\log_b M}{\log_b a}$	

Properties of Logarithms (Recall that logs are only defined for positive values of x.)

For the natural logarithm
1. $\ln xy = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln x^y = y \cdot \ln x$
4. $\ln e^x = x$
5. $e^{\ln x} = x$

For logarithms base a
1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^y = y \cdot \log_a x$
4. $\log_a a^x = x$
5. $a^{\log_a x} = x$

General Properties
1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = 0$
4. $b^{\log_b x} = x$

Useful Identities for Logarithms

For the natural logarithm
1. $\ln e = 1$
2. $\ln 1 = 0$

For logarithms base a
1. $\log_a a = 1$, for all $a > 0$
2. $\log_a 1 = 0$, for all $a > 0$

Natural Logarithms
1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Properties of Natural Logarithms
1. $\ln 1 = 0$ since $e^0 = 1$.
2. $\ln e = 1$ since $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$ inverse property
4. If $\ln x = \ln y$, then $x = y$. one-to-one property

Exponential Laws

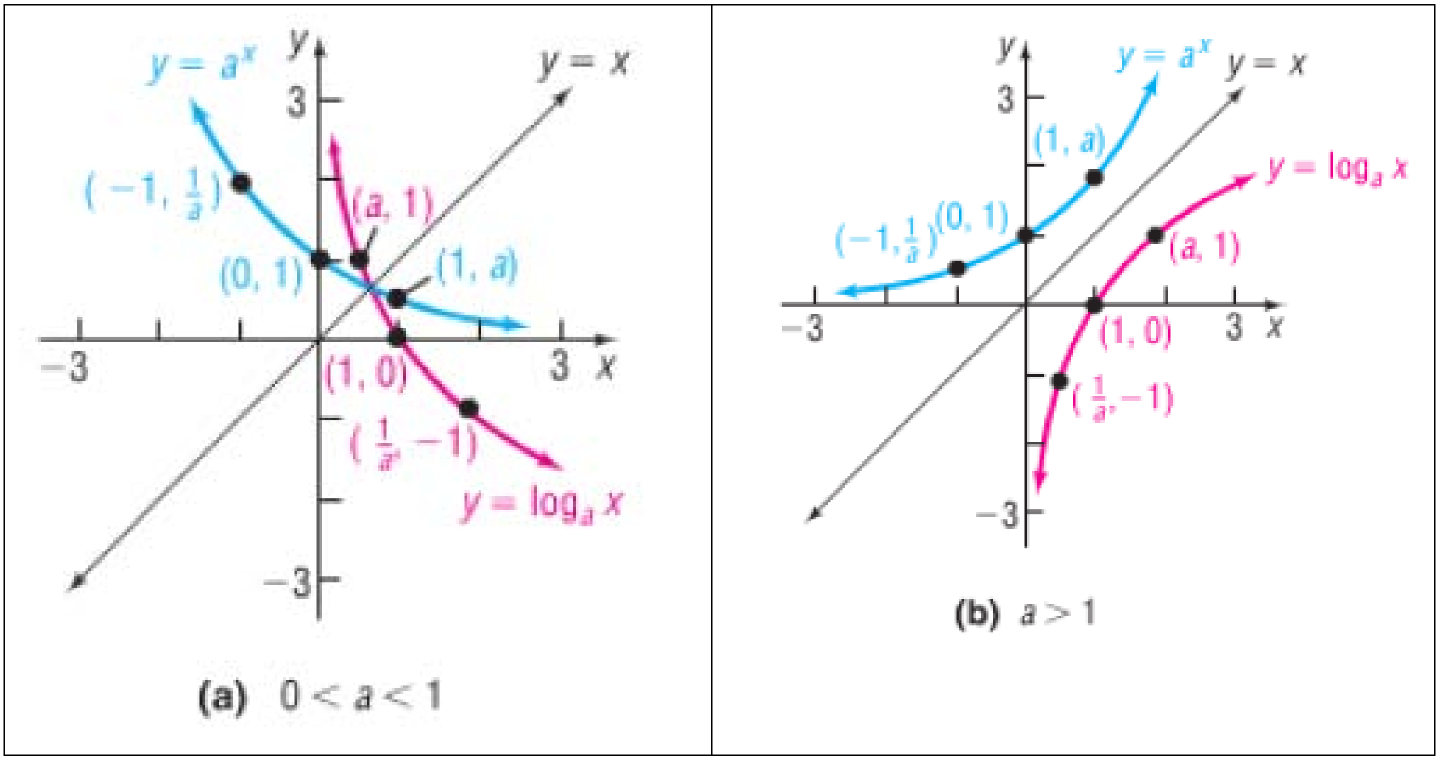
$a^0 = 1$, for $a \neq 0$

$x^1 = x$

$x^0 = 1$

$x^{-1} = 1/x$

$\log_a 1 = 0 \qquad \log_a a = 1$



Example 1

Establishing Properties of Logarithms

(a) Show that $\log_a 1 = 0$. (b) Show that $\log_a a = 1$.

(a) This fact was established when we graphed $y = \log_a x$ (see Figure 30 on page 286). To show the result algebraically, let $y = \log_a 1$. Then

$$\begin{aligned} y &= \log_a 1 \\ a^y &= 1 && \text{Change to an exponential statement.} \\ a^y &= a^0 && a^0 = 1 \text{ since } a > 0,\ a \neq 1 \\ y &= 0 && \text{Solve for } y. \\ \log_a 1 &= 0 && y = \log_a 1 \end{aligned}$$

(b) Let $y = \log_a a$. Then

$$\begin{aligned} y &= \log_a a \\ a^y &= a && \text{Change to an exponential statement.} \\ a^y &= a^1 && a = a^1 \\ y &= 1 && \text{Solve for } y. \\ \log_a a &= 1 && y = \log_a a \end{aligned}$$

Properties of Logarithms

In the properties given next, M and a are positive real numbers, $a \neq 1$, and r is any real number.

The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$a^{\log_a M} = M \qquad \textbf{(1)}$$

The logarithm to the base a of a raised to a power equals that power. That is,

$$\log_a a^r = r \qquad \textbf{(2)}$$

Example 2

Using Properties (1) and (2)

(a) $2^{\log_2 \pi} = \pi$ (b) $\log_{0.2} 0.2^{-\sqrt{2}} = -\sqrt{2}$ (c) $\ln e^{kt} = kt$

The proof uses the fact that $y = a^x$ and $y = \log_a x$ are inverses.

Proof of Property (1) For inverse functions,

$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ in the domain of } f^{-1}$$

Using $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we find

$$f(f^{-1}(x)) = a^{\log_a x} = x \quad \text{for } x > 0$$

Now let $x = M$ to obtain $a^{\log_a M} = M$, where $M > 0$.

Proof of Property (2) For inverse functions,

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in the domain of } f$$

Using $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we find

$$f^{-1}(f(x)) = \log_a a^x = x \quad \text{for all real numbers } x$$

Now let $x = r$ to obtain $\log_a a^r = r$, where r is any real number.

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$$\log_a(MN) = \log_a M + \log_a N \tag{3}$$

The Log of a Quotient Equals the Difference of the Logs

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \tag{4}$$

The Log of a Power Equals the Product of the Power and the Log

$$\log_a M^r = r \log_a M \tag{5}$$

$$a^x = e^{x \ln a} \tag{6}$$

We shall derive properties (3), (5), and (6) and leave the derivation of property (4) as an exercise (see Problem 109).

Proof of Property (3) Let $A = \log_a M$ and let $B = \log_a N$. These expressions are equivalent to the exponential expressions

$$a^A = M \quad \text{and} \quad a^B = N$$

Now

$$\begin{aligned} \log_a(MN) = \log_a(a^A a^B) &= \log_a a^{A+B} && \text{Law of Exponents} \\ &= A + B && \text{Property (2) of logarithms} \\ &= \log_a M + \log_a N \end{aligned}$$

■

Proof of Property (5) Let $A = \log_a M$. This expression is equivalent to

$$a^A = M$$

Now

$$\begin{aligned} \log_a M^r = \log_a(a^A)^r &= \log_a a^{rA} && \text{Law of Exponents} \\ &= rA && \text{Property (2) of logarithms} \\ &= r \log_a M \end{aligned}$$

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Proof of Property (6) From property (1), with $a = e$, we have

$$e^{\ln M} = M$$

Now let $M = a^x$ and apply property (5).

$$e^{\ln a^x} = e^{x \ln a} = a^x$$

■

Write a Logarithmic Expression as a Sum or Difference of Logarithms

Logarithms can be used to transform products into sums, quotients into differences, and powers into factors. Such transformations prove useful in certain types of calculus problems.

Example 3

Writing a Logarithmic Expression as a Sum of Logarithms

Write $\log_a\left(x\sqrt{x^2+1}\right)$, $x > 0$, as a sum of logarithms. Express all powers as factors.

$$\begin{aligned}\log_a\left(x\sqrt{x^2+1}\right) &= \log_a x + \log_a \sqrt{x^2+1} && \log_a(M \cdot N) = \log_a M + \log_a N \\ &= \log_a x + \log_a(x^2+1)^{1/2} \\ &= \log_a x + \frac{1}{2}\log_a(x^2+1) && \log_a M^r = r\log_a M\end{aligned}$$

Example 4

Writing a Logarithmic Expression as a Difference of Logarithms

Write

$$\ln\frac{x^2}{(x-1)^3} \qquad x > 1$$

as a difference of logarithms. Express all powers as factors.

$$\ln\frac{x^2}{(x-1)^3} = \underbrace{\ln x^2 - \ln(x-1)^3}_{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N} = \underbrace{2\ln x - 3\ln(x-1)}_{\log_a M^r = r\log_a M}$$

Example 5

Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write

$$\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} \quad x > 0$$

as a sum and difference of logarithms. Express all powers as factors.

> **WARNING** In using properties (3) through (5), be careful about the values that the variable may assume. For example, the domain of the variable for $\log_a x$ is $x > 0$ and for $\log_a(x-1)$ it is $x > 1$. If we add these functions, the domain is $x > 1$. That is, the equality
>
> $$\log_a x + \log_a(x-1) = \log_a[x(x-1)]$$
>
> is true only for $x > 1$. ■

$$\begin{aligned}
\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} &= \log_a \sqrt{x^2+1} - \log_a[x^3(x+1)^4] && \text{Property (4)} \\
&= \log_a \sqrt{x^2+1} - [\log_a x^3 + \log_a(x+1)^4] && \text{Property (3)} \\
&= \log_a(x^2+1)^{1/2} - \log_a x^3 - \log_a(x+1)^4 \\
&= \frac{1}{2}\log_a(x^2+1) - 3\log_a x - 4\log_a(x+1) && \text{Property (5)}
\end{aligned}$$

Write a Logarithmic Expression as a Single Logarithm

Another use of properties (3) through (5) is to write sums and/or differences of logarithms with the same base as a single logarithm. This skill will be needed to solve certain logarithmic equations discussed in the next section.

Example 6

Writing Expressions as a Single Logarithm

Write each of the following as a single logarithm.

(a) $\log_a 7 + 4\log_a 3$ (b) $\frac{2}{3}\ln 8 - \ln(5^2 - 1)$

(c) $\log_a x + \log_a 9 + \log_a(x^2 + 1) - \log_a 5$

(a)
$$\begin{aligned}\log_a 7 + 4\log_a 3 &= \log_a 7 + \log_a 3^4 && r\log_a M = \log_a M^r\\ &= \log_a 7 + \log_a 81\\ &= \log_a(7\cdot 81) && \log_a M + \log_a N = \log_a(M\cdot N)\\ &= \log_a 567\end{aligned}$$

(b)
$$\begin{aligned}\frac{2}{3}\ln 8 - \ln(5^2 - 1) &= \ln 8^{2/3} - \ln(25 - 1) && r\log_a M = \log_a M^r\\ &= \ln 4 - \ln 24 && 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4\\ &= \ln\left(\frac{4}{24}\right) && \log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)\\ &= \ln\left(\frac{1}{6}\right)\\ &= \ln 1 - \ln 6\\ &= -\ln 6 && \ln 1 = 0\end{aligned}$$

(c)
$$\begin{aligned}\log_a x + \log_a 9 + \log_a(x^2+1) - \log_a 5 &= \log_a(9x) + \log_a(x^2+1) - \log_a 5\\ &= \log_a[9x(x^2 + 1)] - \log_a 5\\ &= \log_a\left[\frac{9x(x^2 + 1)}{5}\right]\end{aligned}$$

WARNING A common error made by some students is to express the logarithm of a sum as the sum of logarithms.

$$\log_a(M + N) \quad \textit{is not equal to} \quad \log_a M + \log_a N$$

Correct statement $\quad \log_a(MN) = \log_a M + \log_a N \qquad$ Property (3)

Another common error is to express the difference of logarithms as the quotient of logarithms.

$$\log_a M - \log_a N \quad \textit{is not equal to} \quad \frac{\log_a M}{\log_a N}$$

Correct statement $\quad \log_a M - \log_a N = \log_a\left(\frac{M}{N}\right) \qquad$ Property (4)

A third common error is to express a logarithm raised to a power as the product of the power times the logarithm.

$$(\log_a M)^r \quad \textit{is not equal to} \quad r \log_a M$$

Correct statement $\quad \log_a M^r = r \log_a M \qquad$ Property (5) ■

57 Two other properties of logarithms that we need to know are consequences of the fact that the logarithmic function $y = \log_a x$ is a one-to-one function.

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$.

$$\text{If } M = N, \text{ then } \log_a M = \log_a N. \qquad \textbf{(7)}$$

$$\text{If } \log_a M = \log_a N, \text{ then } M = N. \qquad \textbf{(8)}$$

When property (7) is used, we start with the equation $M = N$ and say "take the logarithm of both sides" to obtain $\log_a M = \log_a N$.

Properties (7) and (8) are useful for solving *exponential and logarithmic equations*, a topic discussed in the next section.

Example 7

Approximating a Logarithm Whose Base Is Neither 10 Nor e

Approximate $\log_2 7$. Round the answer to four decimal places.

Remember, $\log_2 7$ means "2 raised to what exponent equals 7." If we let $y = \log_2 7$, then $2^y = 7$. Because $2^2 = 4$ and $2^3 = 8$, we expect $\log_2 7$ to be between 2 and 3.

$$\begin{aligned} 2^y &= 7 & & \\ \ln 2^y &= \ln 7 & & \text{Property (7)} \\ y \ln 2 &= \ln 7 & & \text{Property (5)} \\ y &= \frac{\ln 7}{\ln 2} & & \text{Exact value} \\ y &\approx 2.8074 & & \text{Approximate value rounded to four decimal places} \end{aligned}$$

Example 7 shows how to approximate a logarithm whose base is 2 by changing to logarithms involving the base e. In general, we use the **Change-of-Base Formula.**

Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \textbf{(9)}$$

Proof We derive this formula as follows: Let $y = \log_a M$. Then

$$\begin{aligned}
a^y &= M \\
\log_b a^y &= \log_b M && \text{Property (7)} \\
y \log_b a &= \log_b M && \text{Property (5)} \\
y &= \frac{\log_b M}{\log_b a} && \text{Solve for } y. \\
\log_a M &= \frac{\log_b M}{\log_b a} && y = \log_a M
\end{aligned}$$

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Since calculators have keys only for $\boxed{\log}$ and $\boxed{\ln}$, in practice, the Change-of-Base Formula uses either $b = 10$ or $b = e$. That is,

$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a} \qquad \textbf{(10)}$$

Example 8

Using the Change-of-Base Formula

Approximate:

(a) $\log_5 89$ (b) $\log_{\sqrt{2}} \sqrt{5}$

Round answers to four decimal places.

(a) $$\log_5 89 = \frac{\log 89}{\log 5} \approx \frac{1.949390007}{0.6989700043} \approx 2.7889$$

or

$$\log_5 89 = \frac{\ln 89}{\ln 5} \approx \frac{4.48863637}{1.609437912} \approx 2.7889$$

(b) $$\log_{\sqrt{2}} \sqrt{5} = \frac{\log \sqrt{5}}{\log \sqrt{2}} = \frac{\frac{1}{2}\log 5}{\frac{1}{2}\log 2} = \frac{\log 5}{\log 2} \approx 2.3219$$

or

$$\log_{\sqrt{2}} \sqrt{5} = \frac{\ln \sqrt{5}}{\ln \sqrt{2}} = \frac{\frac{1}{2}\ln 5}{\frac{1}{2}\ln 2} = \frac{\ln 5}{\ln 2} \approx 2.3219$$

Historical Feature

John Napier (1550–1617)

Logarithms were invented about 1590 by John Napier (1550–1617) and Joost Bürgi (1552–1632), working independently. Napier, whose work had the greater influence, was a Scottish lord, a secretive man whose neighbors were inclined to believe him to be in league with the devil. His approach to logarithms was very different from ours; it was based on the relationship between arithmetic and geometric sequences, discussed in a later chapter, and not on the inverse function relationship of logarithms to exponential functions (described in Section 5.4). Napier's tables, published in 1614, listed what would now be called *natural logarithms* of sines and were rather difficult to use. A London professor, Henry Briggs, became interested in the tables and visited Napier. In their conversations, they developed the idea of common logarithms, which were published in 1617. Their importance for calculation was immediately recognized, and by 1650 they were being printed as far away as China. They remained an important calculation tool until the advent of the inexpensive handheld calculator about 1972, which has decreased their calculational, but not their theoretical, importance.

A side effect of the invention of logarithms was the popularization of the decimal system of notation for real numbers.