CHAPTER 5 SULLIVAN 9th Ed

5.3 & 5.4 Logarithmic Functions and their graphs

Recall that a one-to-one function $y = f(x)$ has an inverse function that is defined (implicitly) by the equation $x = f(y)$. In particular, the exponential function $y = f(x) = a^x$, where $a > 0$ and $a \neq 1$, is one-to-one and hence has an inverse function that is defined implicitly by the equation

$$x = a^y, \qquad a > 0, \qquad a \neq 1$$

This inverse function is so important that it is given a name, the *logarithmic function*.

The **logarithmic function to the base *a*,** where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

$$\log_a x = y \leftrightarrow x = a^y$$

A logarithm is an exponent!

- **Remember: Logarithmic functions are inverses of exponential functions.**

The inverse of $f(x) = a^x$ is given by $f^{-1}(x) = \log_a x$

As this definition illustrates, **a logarithm is a name for a certain exponent.** So, $\log_a x$ represents the exponent to which a must be raised to obtain x.

Example 1

Relating Logarithms to Exponents

(a) If $y = \log_3 x$, then $x = 3^y$. For example, the logarithmic statement $4 = \log_3 81$ is equivalent to the exponential statement $81 = 3^4$.

(b) If $y = \log_5 x$, then $x = 5^y$. For example, $-1 = \log_5\left(\frac{1}{5}\right)$ is equivalent to $\frac{1}{5} = 5^{-1}$.

Change Exponential Statements to Logarithmic Statements and Logarithmic Statements to Exponential Statements

We can use the definition of a logarithm to convert from exponential form to logarithmic form, and vice versa, as the following two examples illustrate.

Convert each of the following to a logarithmic equation.

a) $25 = 5^x$

$\log_5 25 = x$

b) $e^w = 30$

$\log_e 30 = w$

OTHER EXAMPLES

a) $\log_7 343 = 3$

$\log_7 343 = 3$ $\quad 7^3 = 343$

The logarithm is the exponent.

The base remains the same.

b) $\log_b R = 12$

$\log_b R = 12$ $\quad b^{12} = R$

Example 2

Changing Logarithmic Statements to Exponential Statements

Change each logarithmic statement to an equivalent statement involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

(a) If $\log_a 4 = 5$, then $a^5 = 4$.
(b) If $\log_e b = -3$, then $e^{-3} = b$.
(c) If $\log_3 5 = c$, then $3^c = 5$.

Example 3

Changing Exponential Statements to Logarithmic Statements

Change each exponential statement to an equivalent statement involving a logarithm.

(a) $1.2^3 = m$ (b) $e^b = 9$ (c) $a^4 = 24$

Use the fact that $y = \log_a x$ and $x = a^y$, where $a > 0$ and $a \neq 1$, are equivalent.

(a) If $1.2^3 = m$, then $3 = \log_{1.2} m$.
(b) If $e^b = 9$, then $b = \log_e 9$.
(c) If $a^4 = 24$, then $4 = \log_a 24$.

$y = \log_b x$ $x = b^y$

Logarithmic Equation	Equivalent Exponential Equation	Solution
$y = \log_2 16$	$16 = 2^y$	$16 = 2^4 \rightarrow y = 4$
$y = \log_2(\frac{1}{2})$	$\frac{1}{2} = 2^y$	$\frac{1}{2} = 2^{-1} \rightarrow y = -1$
$y = \log_4 16$	$16 = 4^y$	$16 = 4^2 \rightarrow y = 2$
$y = \log_5 1$	$1 = 5^y$	$1 = 5^0 \rightarrow y = 0$

Evaluate Logarithmic Expressions

To find the exact value of a logarithm, we write the logarithm in exponential notation using the fact that $y = \log_a x$ is equivalent to $a^y = x$ and use the fact that if $a^u = a^v$, then $u = v$.

Example4

Finding the Exact Value of a Logarithmic Expression

Find the exact value of:

(a) $\log_2 16$ (b) $\log_3 \frac{1}{27}$

(a) To evaluate $\log_2 16$, think "2 raised to what power yields 16." So,

$y = \log_2 16$

$2^y = 16$ Change to exponential form.

$2^y = 2^4$ $16 = 2^4$

$y = 4$ Equate exponents.

Therefore, $\log_2 16 = 4$.

(b) To evaluate $\log_3 \frac{1}{27}$, think "3 raised to what power yields $\frac{1}{27}$." So,

$y = \log_3 \frac{1}{27}$

$3^y = \frac{1}{27}$ Change to exponential form.

$3^y = 3^{-3}$ $\frac{1}{27} = \frac{1}{3^3} = 3^{-3}$

$y = -3$ Equate exponents.

Therefore, $\log_3 \frac{1}{27} = -3$.

Determine the Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ has been defined as the inverse of the exponential function $y = a^x$. That is, if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. Based on the discussion given in Section 5.2 on inverse functions, for a function f and its inverse f^{-1}, we have

$$\text{Domain of } f^{-1} = \text{Range of } f \quad \text{and} \quad \text{Range of } f^{-1} = \text{Domain of } f$$

Consequently, it follows that

Domain of the logarithmic function = Range of the exponential function = $(0, \infty)$

Range of the logarithmic function = Domain of the exponential function = $(-\infty, \infty)$

In the next box, we summarize some properties of the logarithmic function:

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $0 < x < \infty$ Range: $-\infty < y < \infty$

The domain of a logarithmic function consists of the *positive* real numbers, so the argument of a logarithmic function must be greater than zero.

Example 5

Finding the Domain of a Logarithmic Function

Find the domain of each logarithmic function.

(a) $F(x) = \log_2(x + 3)$ (b) $g(x) = \log_5\left(\frac{1 + x}{1 - x}\right)$ (c) $h(x) = \log_{1/2}|x|$

(a) The domain of F consists of all x for which $x + 3 > 0$, that is, $x > -3$. Using interval notation, the domain of f is $(-3, \infty)$.

(b) The domain of g is restricted to

$$\frac{1 + x}{1 - x} > 0$$

Solving this inequality, we find that the domain of g consists of all x between -1 and 1, that is, $-1 < x < 1$ or, using interval notation, $(-1, 1)$.

GRAPHS OF LOGARITHMIC FUNCTIONS

Facts about the Graph of a Logarithmic Function $f(x) = \log_b x$

1. The x-intercept of the graph is 1. There is no y-intercept.
2. The y-axis is a vertical asymptote of the graph.
3. A logarithmic function is decreasing if $0 < b < 1$ and increasing if $b > 1$.
4. The graph is smooth and continuous, with no corners or gaps.

$y = \log_b x$ has the following properties

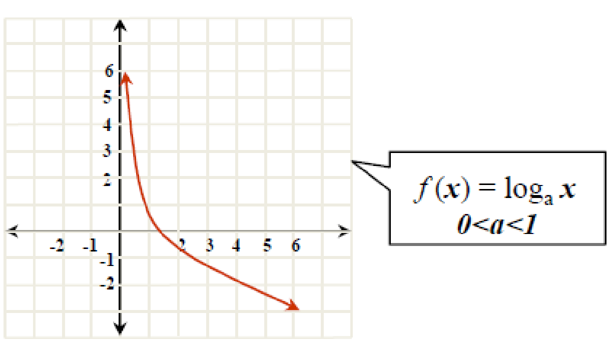
- Domain $(0, \infty)$, Range $(-\infty, \infty)$
- It passes through the point (1,0)
- It passes through the point (b, 1)
- The y- axis is an asymptote.
- If $b > 1$, it is an increasing function
- If $0 < b < 1$, it is a decreasing function

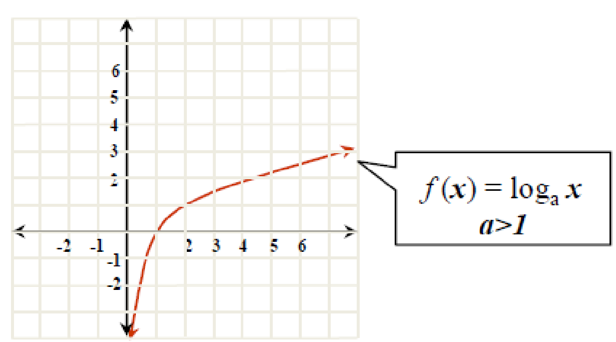
Transformations Involving Logarithmic Functions

Vertical Translation	$f(x) = c + \log_b x$ $f(x) = -c + \log_b x$	Up c units Down c units
Horizontal Translation	$f(x) = \log_b (x + c)$ $f(x) = \log_b (x - c)$	Left c units Right c units
Stretching: Vertical Horizontal	 $f(x) = c \log_b x$ $f(x) = \log_b cx$	 Stretches by c Stretches by 1/c
Reflection	$f(x) = \log_b (-x)$ $f(x) = -\log_b x$	About y-axis About x-axis

Characteristics of the Graphs of Logarithmic Functions of the Form $f(x) = \log_a x$

- The x-intercept is 1. There is no y-intercept.
- The y-axis is a vertical asymptote. (x = 0)
- If 0 < a < 1, the function is decreasing. If b > 1, the function is increasing.
- The graph is smooth and continuous. It has no sharp corners or edges.





LOGARITHMIC FUNCTION $f(x) = \log_a x, \quad a > 1$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

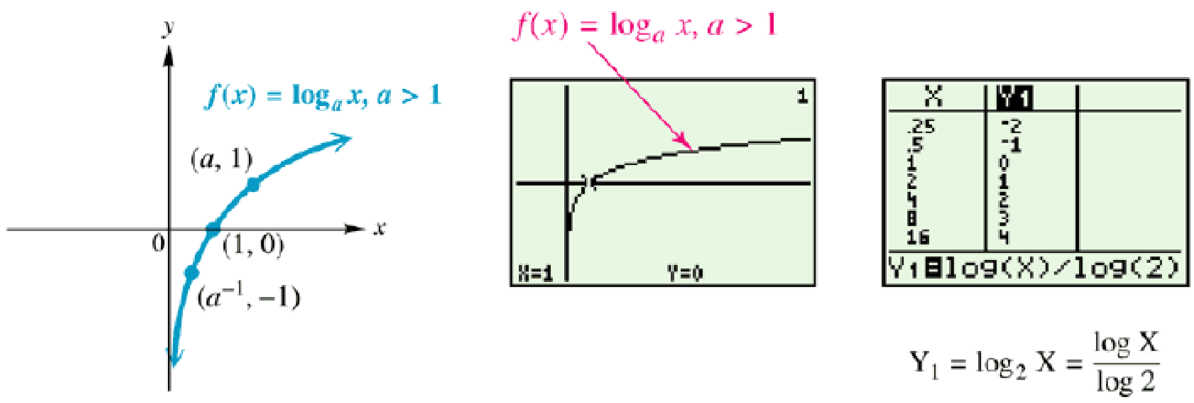


FIGURE 36

- $f(x) = \log_a x$, $a > 1$, is increasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph goes through the points $(a^{-1}, -1)$, $(1, 0)$, and $(a, 1)$.

LOGARITHMIC FUNCTION $f(x) = \log_a x, \quad 0 < a < 1$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

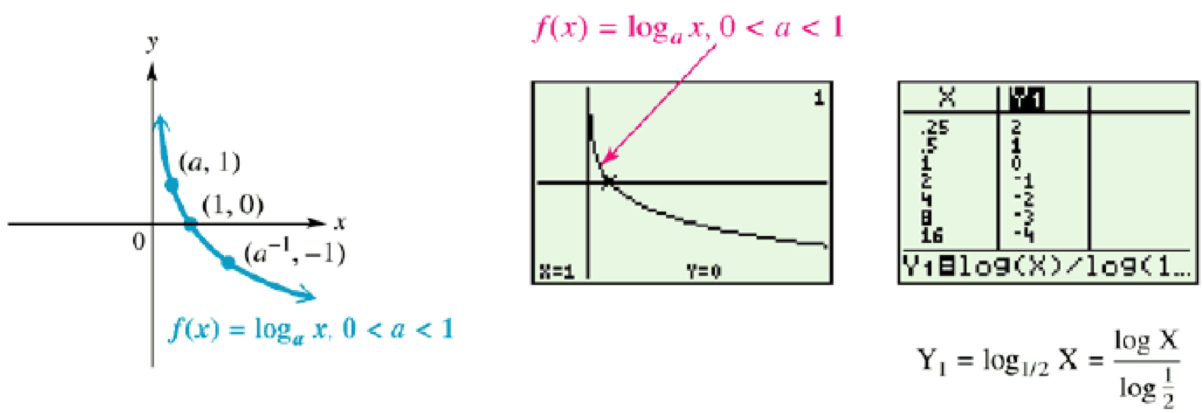
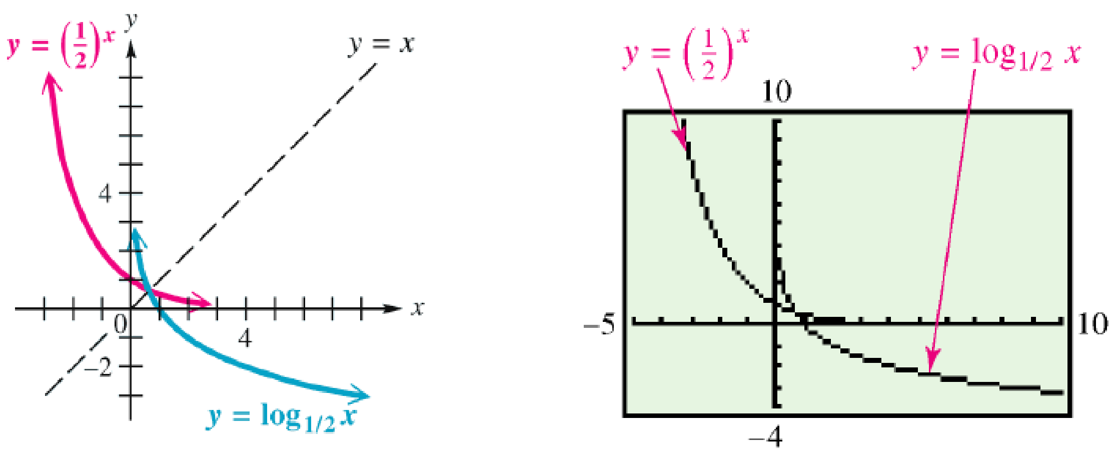


FIGURE 37

- $f(x) = \log_a x$, $0 < a < 1$, is decreasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph goes through the points $(a, 1)$, $(1, 0)$, and $(a^{-1}, -1)$.

Graphs Logs Func 0<a<1

- Below are typical shapes for such graphs where $0 < a < 1$



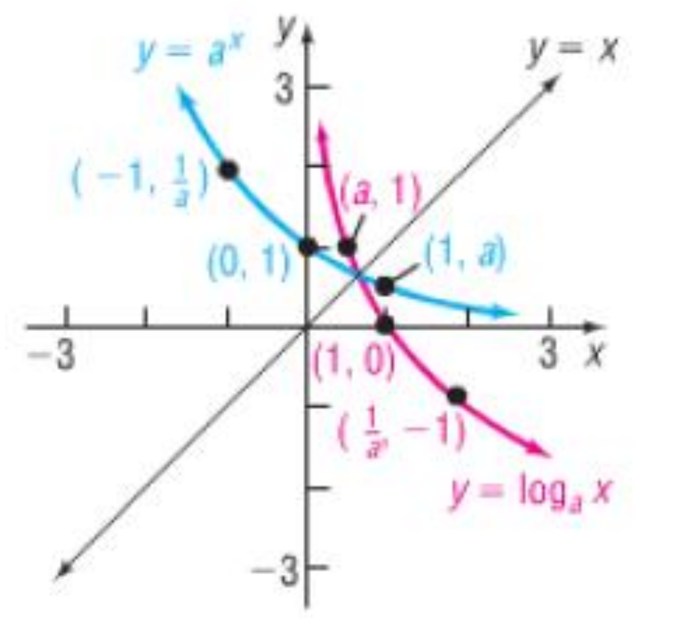
Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain is the set of positive real numbers or $(0, \infty)$ using interval notation; the range is the set of all real numbers or $(-\infty, \infty)$ using interval notation.
2. The x-intercept of the graph is 1. There is no y-intercept.
3. The y-axis ($x = 0$) is a vertical asymptote of the graph.
4. A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
6. The graph is smooth and continuous, with no corners or gaps.

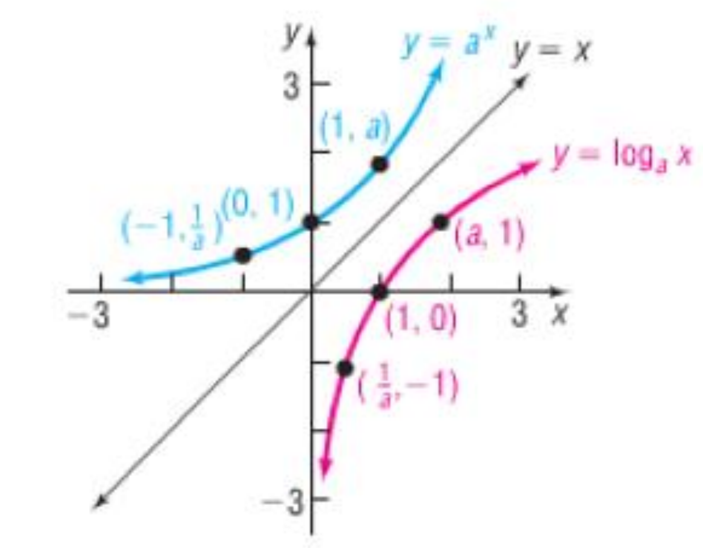
The graphs of $y = \log_a x$ in Figures 30(a) and (b) lead to the following properties.

Since exponential functions and logarithmic functions are inverses of each other, the graph of the logarithmic function $y = \log_a x$ is the reflection about the line $y = x$ of the graph of the exponential function $y = a^x$, as shown in Figure 30.

Figure 30



(a) $0 < a < 1$



(b) $a > 1$

Base is a decimal or fraction. Decreasing function

$Y=a^x$	$y = \log_a x$
Y Intercept = (0,1)	Na
X intercept = n/a	(1,0)

Base is a whole integer greater than 1. Increasing function

$Y=a^x$	$y = \log_a x$

For example, to graph $y = \log_2 x$, graph $y = 2^x$ and reflect it about the line $y = x$. See Figure 31. To graph $y = \log_{1/3} x$, graph $y = \left(\frac{1}{3}\right)^x$ and reflect it about the line $y = x$. See Figure 32.

Figure 31

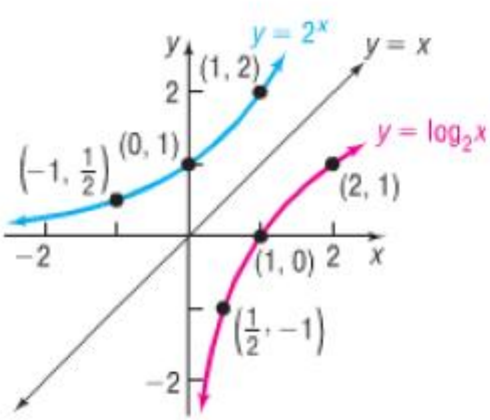
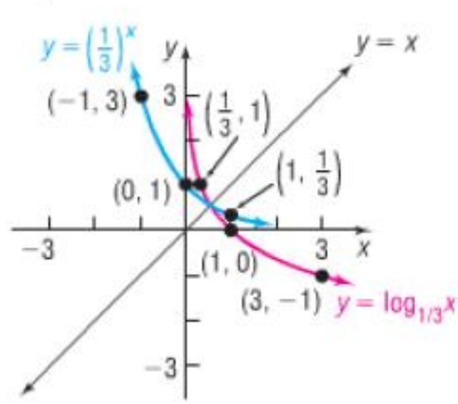


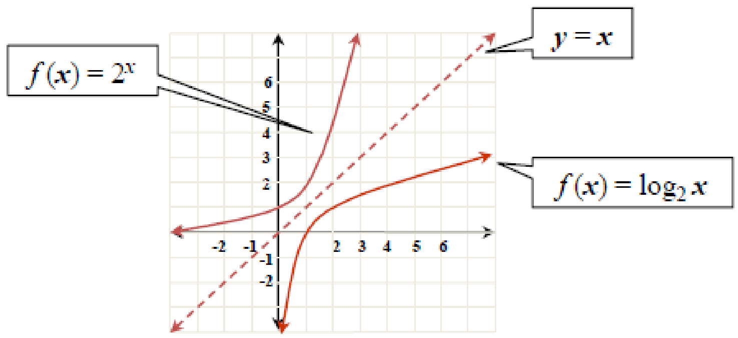
Figure 32



Graph $f(x) = 2^x$ and $g(x) = \log_2 x$ in the same rectangular coordinate system.

Solution

We now sketch the basic exponential graph. The graph of the inverse (logarithmic) can also be drawn by reflecting the graph of $f(x) = 2^x$ over the line y = x.



Graph $f(x) = 2^x$ and $g(x) = \log_2 x$ in the same rectangular coordinate system.

Solution We first set up a table of coordinates for $f(x) = 2^x$. Reversing these coordinates gives the coordinates for the inverse function, $g(x) = \log_2 x$.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	1/4	1/2	1	2	4	8

x	1/4	1/2	1	2	4	8
$g(x) = \log_2 x$	-2	-1	0	1	2	3

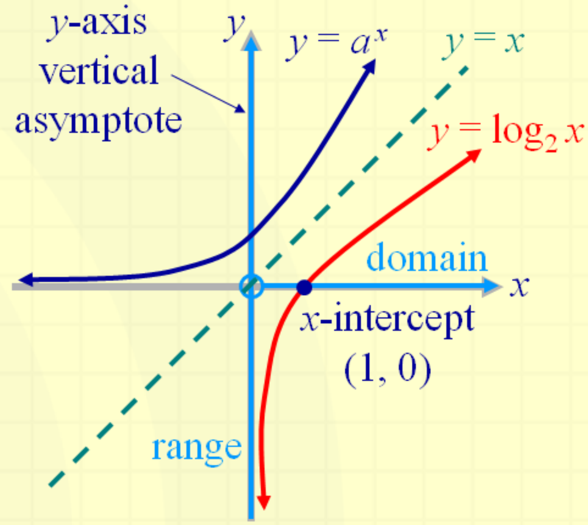
Reverse coordinates.

The graphs of logarithmic functions are similar for different values of a.

$$f(x) = \log_a x \quad (a > 1)$$

Graph of $f(x) = \log_a x \quad (a > 1)$

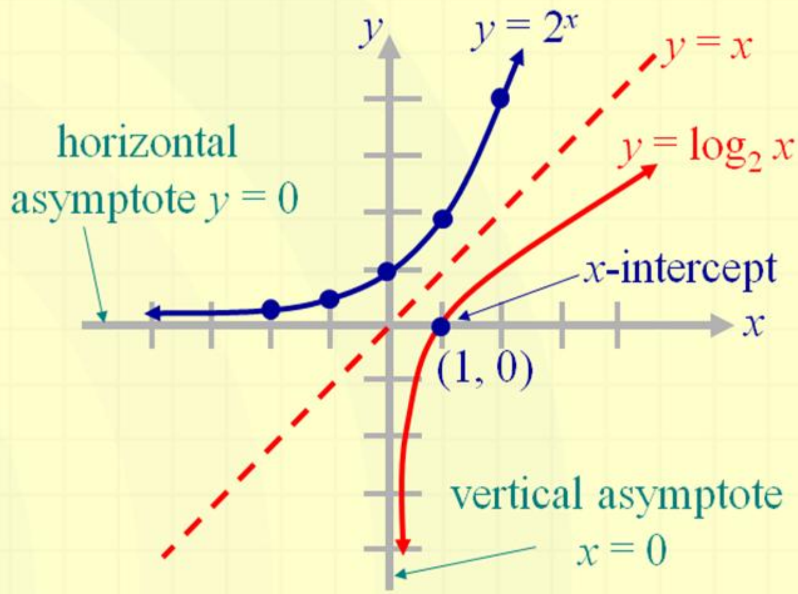
1. domain $(0, \infty)$
2. range $(-\infty, +\infty)$
3. x-intercept $(1, 0)$
4. vertical asymptote
   $x = 0$ as $x \to 0^+$ $f(x) \to -\infty$
5. increasing
6. continuous
7. one-to-one
8. reflection of $y = a^x$ in $y = x$



Graph $f(x) = \log_2 x$

Since the logarithm function is the *inverse* of the exponential function of the same base, its graph is the reflection of the exponential function in the line $y = x$.

x	2^x
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



If the base of a logarithmic function is the number e, then we have the **natural logarithm function.** This function occurs so frequently in applications that it is given a special symbol, **ln** (from the Latin, *logarithmus naturalis*). That is,

$$y = \ln x \quad \text{if and only if} \quad x = e^y \qquad \textbf{(1)}$$

$y = \log_e x$ is written $y = \ln x$

Summary Logs Base "e" and ln

- $\log_e x$ means ln x
- These are called natural logarithms
- y = ln x is the inverse of y = e^x
- The domain of y = ln x is (0, ∞)
- The range is the interval (-∞, ∞)

Since $y = \ln x$ and the exponential function $y = e^x$ are inverse functions, we can obtain the graph of $y = \ln x$ by reflecting the graph of $y = e^x$ about the line $y = x$. See Figure 33.

Using a calculator with an ln key, we can obtain other points on the graph of $f(x) = \ln x$. See Table 7.

Figure 33

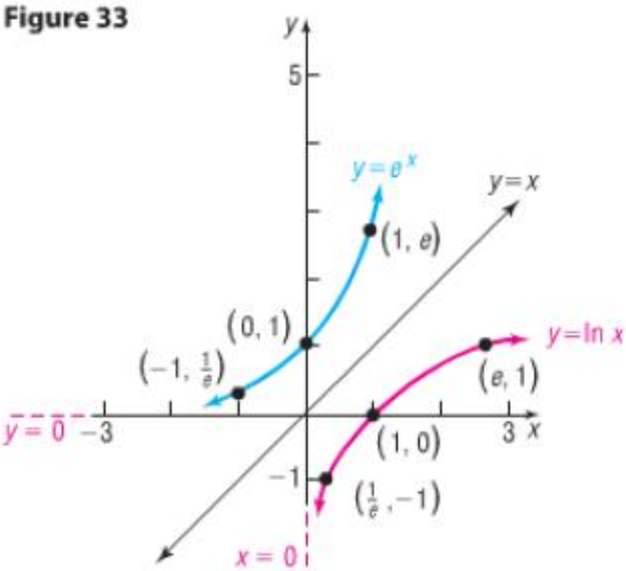


Table 7

x	ln x
$\frac{1}{2}$	−0.69
2	0.69
3	1.10

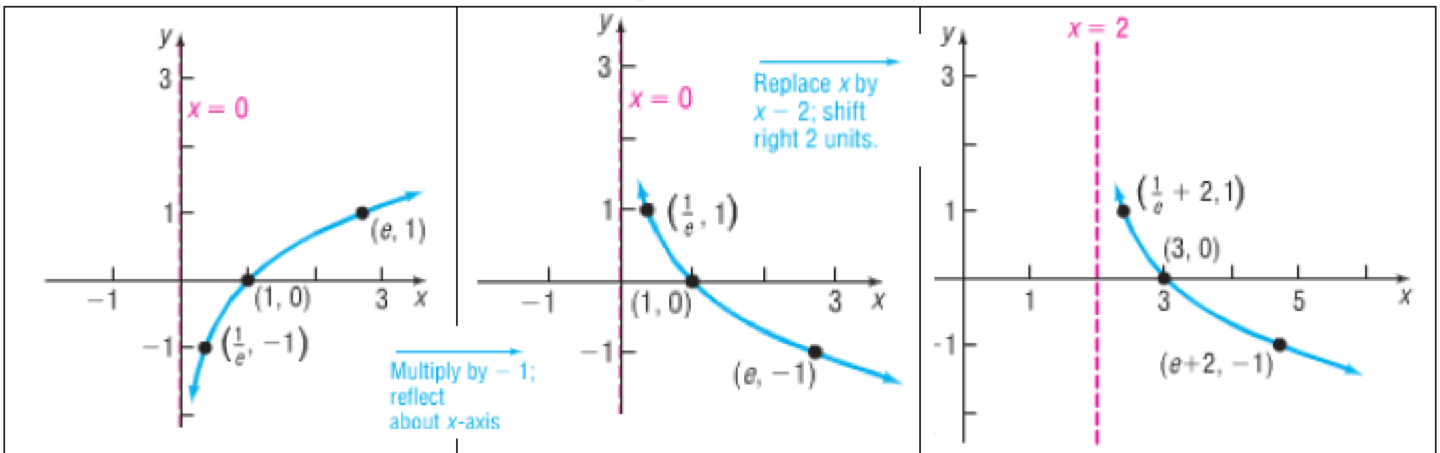
Example 6

Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = -\ln(x - 2)$.
(b) Graph f.
(c) From the graph, determine the range and vertical asymptote of f.
(d) Find f^{-1}, the inverse of f.
(e) Find the domain and the range of f^{-1}.
(f) Graph f^{-1}.

(a) The domain of f consists of all x for which $x - 2 > 0$ or, equivalently, $x > 2$. The domain of f is $\{x | x > 2\}$ or $(2, \infty)$ in interval notation.

(b) To obtain the graph of $y = -\ln(x - 2)$, we begin with the graph of $y = \ln x$ and use transformations. See Figure 34.



(a) $y = \ln x$ (b) $y = -\ln x$ (c) $y = -\ln(x - 2)$

(c) The range of $f(x) = -\ln(x - 2)$ is the set of all real numbers. The vertical asymptote is $x = 2$. [Do you see why? The original asymptote $(x = 0)$ is shifted to the right 2 units.]

(d) To find f^{-1}, begin with $y = -\ln(x - 2)$. The inverse function is defined (implicitly) by the equation

$$x = -\ln(y - 2)$$

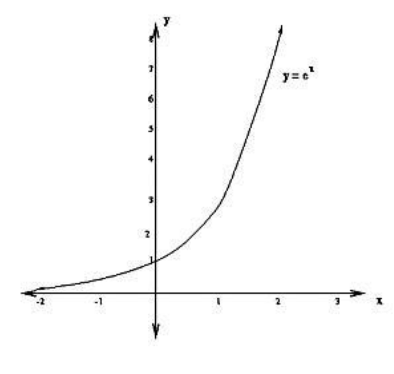
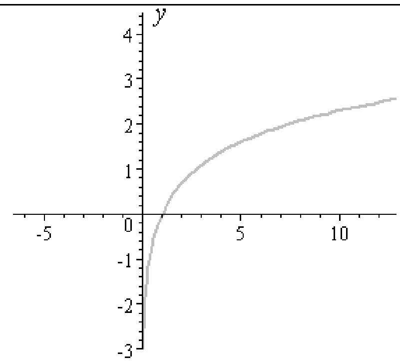
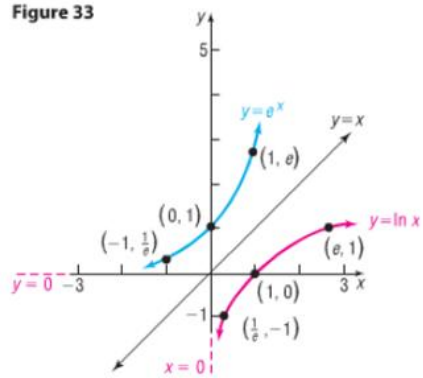
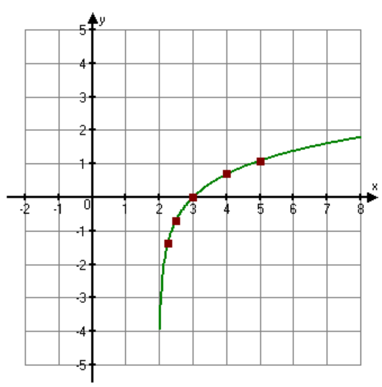
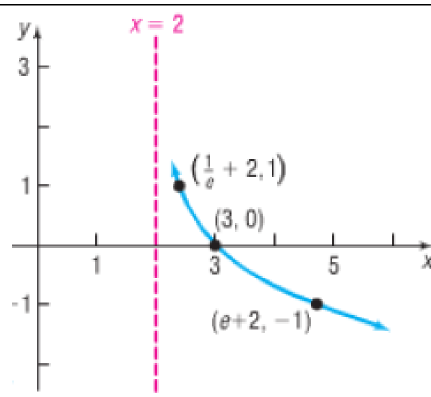
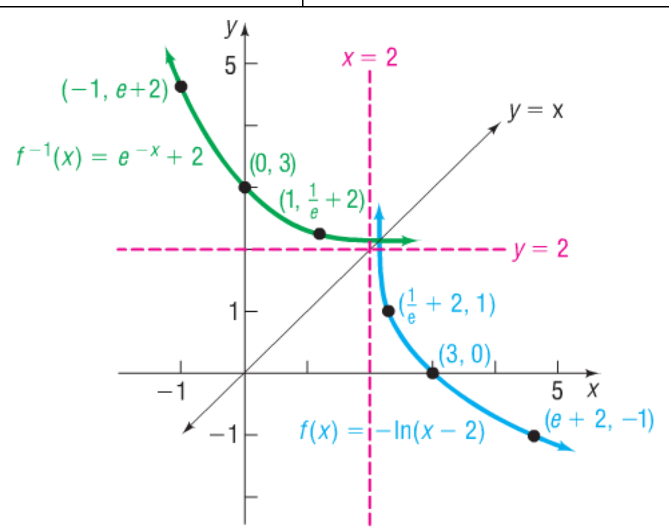
Proceed to solve for y.

$$-x = \ln(y - 2) \quad \text{Isolate the logarithm.}$$
$$e^{-x} = y - 2 \quad \text{Change to an exponential statement.}$$
$$y = e^{-x} + 2 \quad \text{Solve for } y.$$

The inverse of f is $f^{-1}(x) = e^{-x} + 2$.

(e) The domain of f^{-1} equals the range of f, which is the set of all real numbers, from part (c). The range of f^{-1} is the domain of f, which is $(2, \infty)$ in interval notation.

(f) To graph f^{-1}, use the graph of f in Figure 34(c) and reflect it about the line $y = x$. See Figure 35. We could also graph $f^{-1}(x) = e^{-x} + 2$ using transformations.

		Figure 33 
Y =e^x	**y = lnx**	**Y =e^x and y = lnx**
		
Y = ln (x-2)	$y = -\ln(x-2)$	
Y =e^x [Exponential Function] **y = lnx** [Natural log, inverse of exp func] $y = -\ln(x-2)$ [natural log reflected over x-axis] $f^{-1}(x) = e^{-x} + 2$ [Inverse of $y = -\ln(x-2)$]		

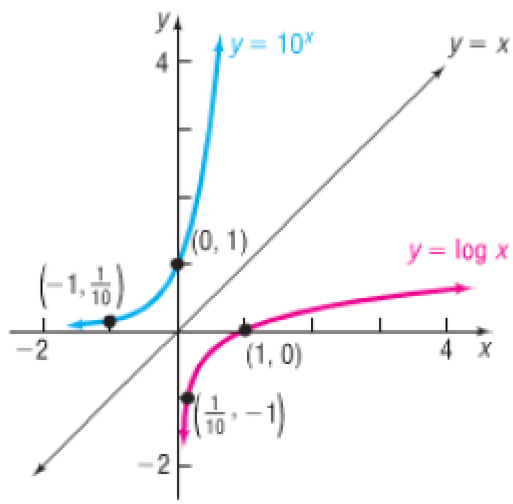
Graphs of Logs

If the base of a logarithmic function is the number 10, then we have the **common logarithm function.** If the base a of the logarithmic function is not indicated, it is understood to be 10. That is,

$$y = \log x \quad \text{if and only if} \quad x = 10^y$$

Since $y = \log x$ and the exponential function $y = 10^x$ are inverse functions, we can obtain the graph of $y = \log x$ by reflecting the graph of $y = 10^x$ about the line $y = x$. See Figure 36.

Figure 36



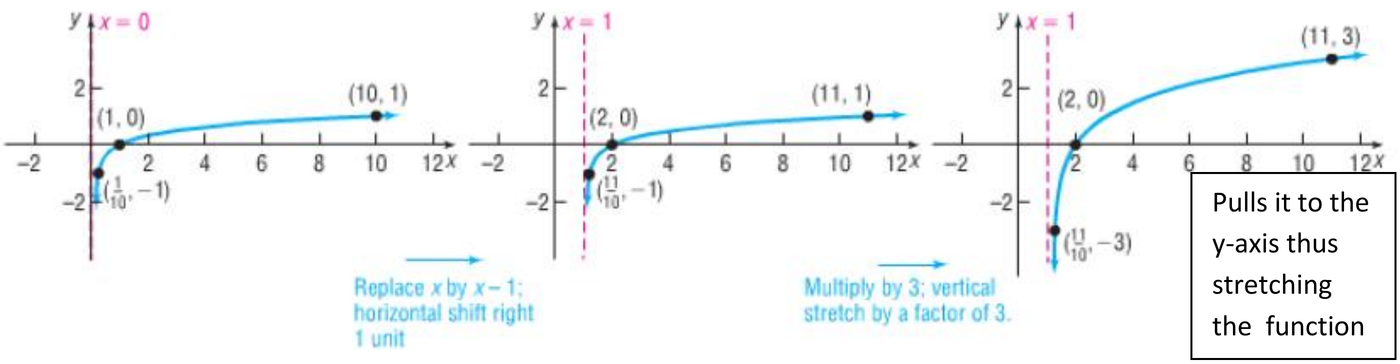
Example 7

Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = 3\log(x - 1)$.
(b) Graph f.
(c) From the graph, determine the range and vertical asymptote of f.
(d) Find f^{-1}, the inverse of f.
(e) Find the domain and the range of f^{-1}.
(f) Graph f^{-1}.

(a) The domain of f consists of all x for which $x - 1 > 0$ or, equivalently, $x > 1$. The domain of f is $\{x | x > 1\}$ or $(1, \infty)$ in interval notation.

(b) To obtain the graph of $y = 3\log(x - 1)$, begin with the graph of $y = \log x$ and use transformations. See Figure 37.



(a) $y = \log x$ **(b)** $y = \log(x - 1)$ **(c)** $y = 3\log(x - 1)$

(c) The range of $f(x) = 3\log(x - 1)$ is the set of all real numbers. The vertical asymptote is $x = 1$.

(d) Begin with $y = 3\log(x - 1)$. The inverse function is defined (implicitly) by the equation

$$x = 3\log(y - 1)$$

Proceed to solve for y.

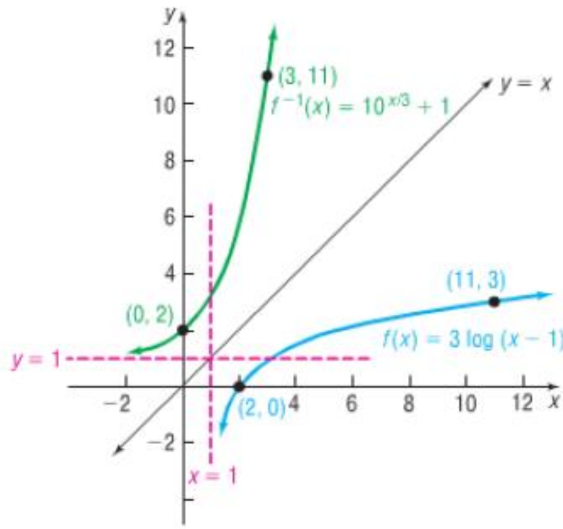
$$\frac{x}{3} = \log(y - 1) \quad \text{Isolate the logarithm.}$$
$$10^{x/3} = y - 1 \quad \text{Change to an exponential statement.}$$
$$y = 10^{x/3} + 1 \quad \text{Solve for } y.$$

The inverse of f is $f^{-1}(x) = 10^{x/3} + 1$.

(e) The domain of f^{-1} is the range of f, which is the set of all real numbers, from part (c). The range of f^{-1} is the domain of f, which is $(1, \infty)$ in interval notation.

(f) To graph f^{-1}, we use the graph of f in Figure 37(c) and reflect it about the line $y = x$. See Figure 38. We could also graph $f^{-1}(x) = 10^{x/3} + 1$ using transformations.

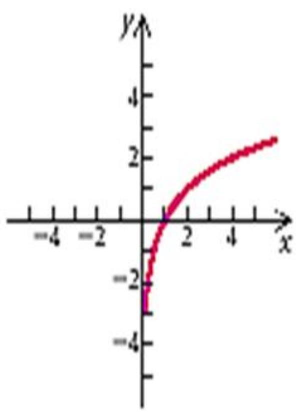
Figure 38



Comparing Exponential and Logarithmic Functions

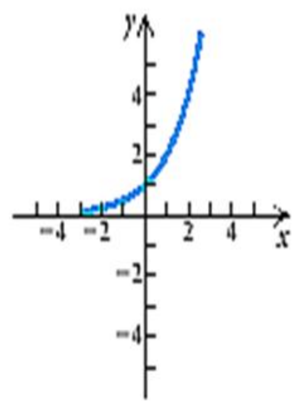
LOGARITHMIC FUNCTION

$x = a^y$
$f^{-1}(x) = \log_a x$
$a > 1$
Continuous
One-to-one
Domain: All positive real numbers, $(0, \infty)$
Range: All real numbers, $(-\infty, \infty)$
Increasing
Vertical asymptote is y-axis: ($\log_a x \to -\infty$ as $x \to 0^+$)
x-intercept: $(1, 0)$
There is no y-intercept.



EXPONENTIAL FUNCTION

$y = a^x$
$f(x) = a^x$
$a > 1$
Continuous
One-to-one
Domain: All real numbers, $(-\infty, \infty)$
Range: All positive real numbers, $(0, \infty)$
Increasing
Horizontal asymptote is x-axis: ($a^x \to 0$ as $x \to -\infty$)
y-intercept: $(0, 1)$
There is no x-intercept.



Example 8

Solving Logarithmic Equations

Solve:

(a) $\log_3(4x - 7) = 2$ (b) $\log_x 64 = 2$

(a) We can obtain an exact solution by changing the logarithmic equation to exponential form.

$$\log_3(4x - 7) = 2$$
$$4x - 7 = 3^2 \quad \text{Change to exponential form using } y = \log_a x \text{ means } a^y = x.$$
$$4x - 7 = 9$$
$$4x = 16$$
$$x = 4$$

✓**Check:** $\log_3(4x - 7) = \log_3(4 \cdot 4 - 7) = \log_3 9 = 2$ $3^2 = 9$

The solution set is $\{4\}$.

(b) We can obtain an exact solution by changing the logarithmic equation to exponential form.

$$\log_x 64 = 2$$
$$x^2 = 64 \quad \text{Change to exponential form.}$$
$$x = \pm\sqrt{64} = \pm 8 \quad \text{Square Root Method}$$

The base of a logarithm is always positive. As a result, we discard -8. We check the solution 8.

✓**Check:** $\log_8 64 = 2$ $8^2 = 64$

The solution set is $\{8\}$.

Example 9

Using Logarithms to Solve an Exponential Equation

Solve: $e^{2x} = 5$

We can obtain an exact solution by changing the exponential equation to logarithmic form.

$$e^{2x} = 5$$

$$\ln 5 = 2x \quad \text{Change to logarithmic form using the fact that if } e^y = x \text{ then } y = \ln x.$$

$$x = \frac{\ln 5}{2} \quad \text{Exact solution}$$

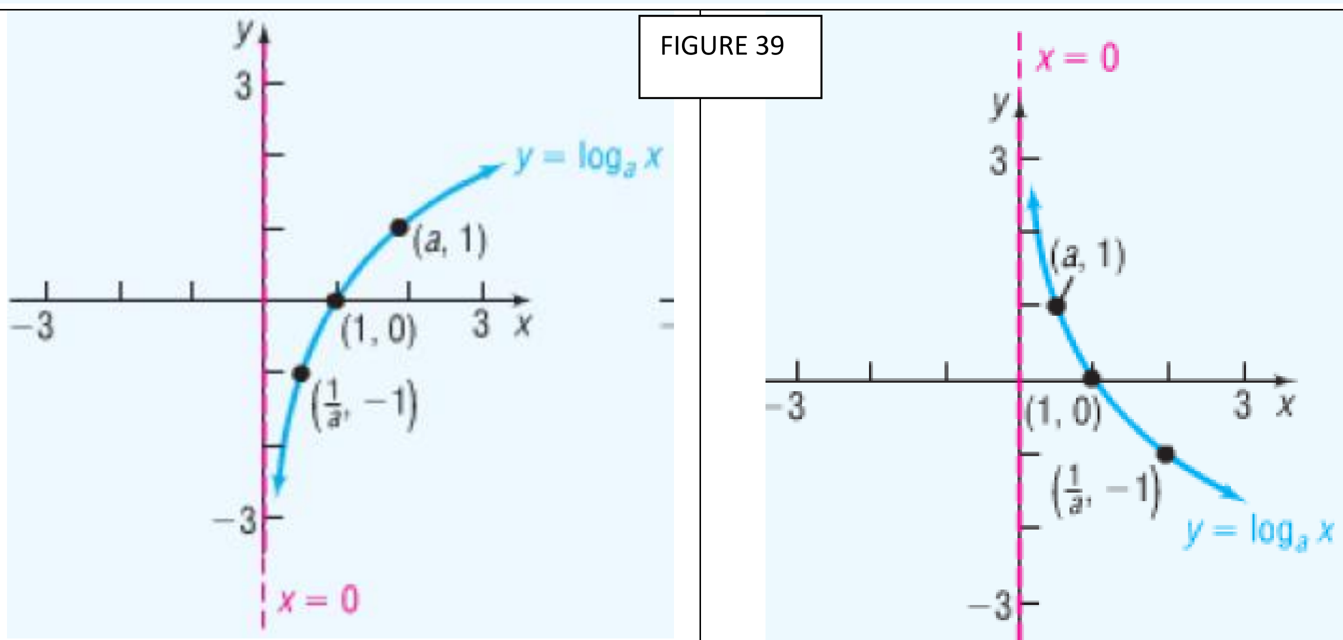
$$\approx 0.805 \quad \text{Approximate solution}$$

The solution set is $\left\{\frac{\ln 5}{2}\right\}$.

PROPERTIES OF LOGARITHMIC FUNCTIONS

$f(x) = \log_a x, \quad a > 1$	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$
($y = \log_a x$ means $x = a^y$)	x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one
	See Figure 39(a) for a typical graph.
$f(x) = \log_a x, \quad 0 < a < 1$	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$
($y = \log_a x$ means $x = a^y$)	x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one
	See Figure 39(b) for a typical graph.

FIGURE 39



(a) $a > 1$ example $y = \log_2 x$ | (b) $0 < a < 1$ example $y = \log_{1/2} x$